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RESEARCH IN AND APPLICATION OF MODERN AUTOMATIC CONTROL THEORY TO NUCLEAR ROCKET DYNAMICS AND CONTROL

Semi-Annual Status Report No. 1

Prepared Under Grant NsG-490

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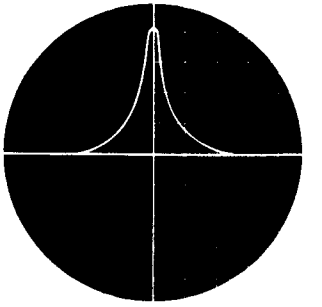

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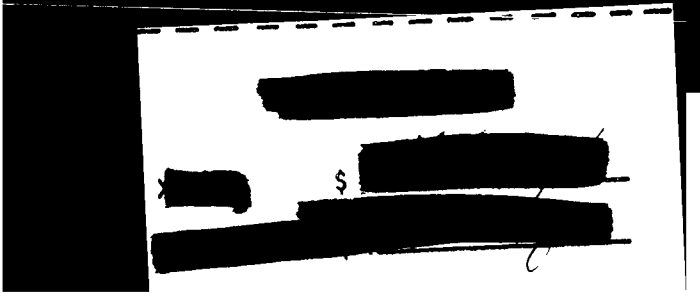

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RESEARCH IN AND APPLICATION OF MODERN AUTOMATIC CONTROL THEORY TO NUCLEAR ROCKET DYNAMICS AND CONTROL

Nuclear Engineering Department
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INTRODUCTION

The research progress given in this report represents that accomplished over the first six months of work on the project. The effective date of the grant was July 1, 1963; however, work did not begin until September 1, 1963 when the majority of the faculty and graduate students returned to the University. The direction this research will take during the next report period is altered somewhat from that reported here. The decision to modify this direction came after faculty working on the research project received security clearances and were permitted to obtain classified information on the ROVER Project, from the staff of the Los Alamos Scientific Laboratory. This information was not available to the researchers working on the grant until well into the first report period. The research to be conducted during the second report period will be more pertinent to particular problems being encountered in the development of the nuclear rocket.

This progress report is divided into two parts. Part I deals with the application of the Second Method of Liapunov to stability problems in nuclear reactors. Several examples related to the nuclear rocket engine are given. At the end of this section an outline of the research to be conducted during the next report period is presented. Part II of the report is concerned with the application of Pontryagin's Maximum Principle to the solution of optimization problems in nuclear reactors. An optimum shutdown program is determined for reactors, which will minimize the buildup of xenon poisoning after shutdown. Solution of this problem has given some insight into the problem of determining an optimum program for propellant release that will minimize the amount of propellant needed to accomodate the after-

shutdown heat generation in nuclear rocket engines. An outline of the research to be carried on in the application of the Maximum Principle to nuclear rocket problems is given in the latter part of the section.

The results of the research given in this report will be presented in two papers ~~to~~ be given at the American Nuclear Society Meeting ~~in~~ Philadelphia ~~in~~, June 1964.

PART I

The Application of the Second Method of Liapunov to Non-Linear Stability Problems in Nuclear Reactors

PART I

I. Introduction

The Second Method of Liapunov provides a general approach to the stability of dynamic systems described by ordinary linear or nonlinear differential equations. This fact has been recognized by a small group of nuclear engineers, and application of the Second Method has been made to the inherently nonlinear reactor kinetic equations. However, as yet, the Second Method has not attained the prominence among nuclear engineers that it enjoys in the other engineering disciplines, particularly in the area of automatic control.

Five years ago this was understandable, as the principal theorems of Liapunov were still largely scattered in the Russian literature, or known only to a relatively small group of mathematicians. However, this is no longer the case; and, indeed, significant advances have been made by American authors in applying the Second Method of Liapunov to practical engineering problems.

II. Introduction to the Second Method

In this report, the dynamic systems under consideration are assumed to be autonomous and describable in state variable form as n first-order differential equations (1).

$$\begin{aligned} \dot{x}_1 &= b_{11}(\underline{x})\, x_1 + b_{12}(\underline{x})\, x_2 + \ldots + b_{1n}(\underline{x})\, x_n \\ &\vdots \\ \dot{x}_n &= b_{n1}(\underline{x})\, x_1 + \ldots \qquad\qquad + b_{nn}(\underline{x})\, x_n \end{aligned} \tag{1}$$

where n is the order of the system and $\underline{x}$ is an n-vector (column vector) representing the state of the system. In matrix notation, this may be written compactly as:

$$\dot{\underline{x}} = B(\underline{x})\ \underline{x} \tag{2}$$

where:

$$B(\underline{x}) = \begin{pmatrix} b_{11}(\underline{x}) & \cdots & b_{1n}(\underline{x}) \\ \vdots & & \\ b_{n1}(\underline{x}) & \cdots & b_{nn}(\underline{x}) \end{pmatrix} \tag{3}$$

The $b_{ij}(\underline{x})$ may be any continuous function of the state variables x_k.

The equilibrium state being investigated must be located at the origin. This is actually no restriction, since any equilibrium point may always be translated by simple linear change of variables to the origin. Let $\| \underline{x} \|$ be the Euclidean length of the vector $\underline{x}$, i.e. $\| x \|^2 = x_1^2 + \ldots + x_n^2$, and S(R) be a spherical region of radius $R>0$ around the origin, i.e. S(R) consists of all points $\underline{x}$ satisfying $\| \underline{x} \| \leq R$.

> Definition of Asymptotic Stability: The origin is asymptotically stable if corresponding to each S(R) there is an S(r) such that solutions starting in S(r) do not leave S(R) but approach the origin as $t \longrightarrow \infty$.

Only asymptotic stability will be considered in this report. Simply stated, an autonomous system is asymptotically stable if it returns to its original equilibrium state. If the definition holds in the whole space, the system is globally asymptotically stable.

The following modified* Liapunov stability theorem, due to LaSalle, can now be stated:

---

* This theorem differs from the original Liapunov theorem in condition (b) where V(x) is allowed to be equal or less than zero as long as it is not zero on a solution of the system, other than the trivial solution, $\underline{x} = 0$.

A Modified Liapunov Stability Theorem: If there exists a scalar function $V(\underline{x})$ with continuous first partials such that

a. $V(\underline{x}) > 0$ for all $x \neq 0$; $V(\underline{0}) = 0$ (positive definite)

b. $\dot{V}(x) \leq 0$ for all $\underline{x}$ (at least negative semidefinite)

c. $V(x) \longrightarrow \infty$ as $\| x \| \longrightarrow \infty$.

then, if $\dot{V}$ is not identically zero along any solution of (2) other than the origin, the system is globally asymptotically stable.

Since $V(x)$ has continuous first partials, the chain rule may be used to determine $\dot{V}(x)$

$$\dot{V} = \frac{dV}{dt} = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \cdots \frac{\partial V}{\partial x_n}\frac{dx_n}{dt}$$
$$= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 + \cdots + \frac{\partial V}{\partial x_n}\dot{x}_n$$

Then defining ∇V, the gradient of V, as the column vector

$$\nabla V = \begin{pmatrix} \frac{\partial V}{\partial x_1} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{pmatrix}$$

, $\dot{V}$ may be written in matrix form as

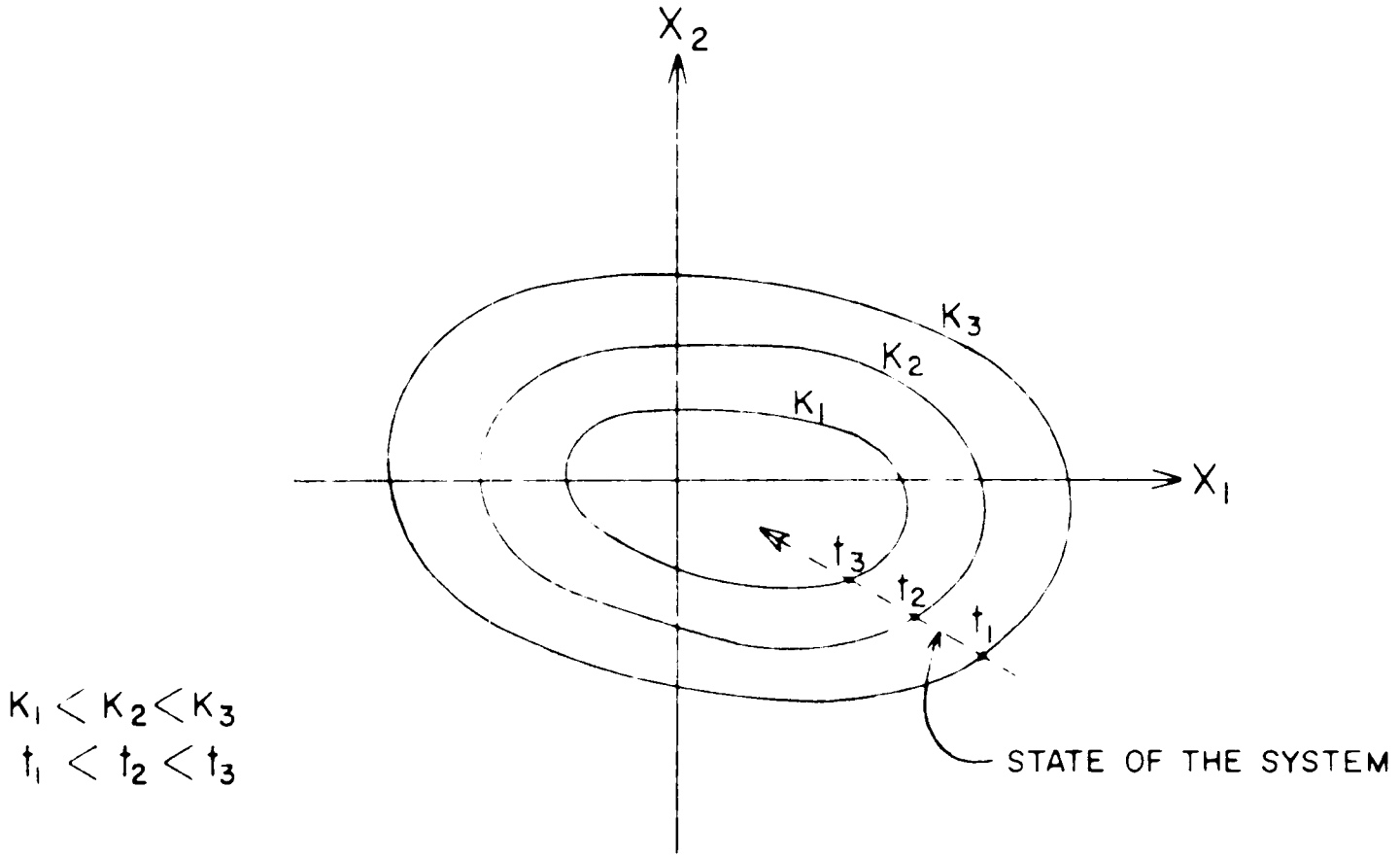
$$\dot{V} = \nabla V' \dot{x} \qquad (4)$$

The basic concept of the Second Method is now evident: by proper selection or generation of a Liapunov V-function, it is possible to determine the stability of a nonlinear dynamic system without any knowledge of

the solution of the system equation. A technique for determining V-functions is presented later. Before proceeding to the applications of the Second Method, it is perhaps of value to investigate the stability theorem from a geometric viewpoint.

By requiring $V > 0$ for $x \neq 0$, $V(x) = \text{const}$ becomes a family of concentric closed surfaces surrounding the origin such that the surface $V(x) = K_1$, lies inside $V(x) = K_2$ whenever $K_1 < K_2$. Figure 1 shows a graphical picture for the two dimensional or second order case. Since both V and $\dot{V}$ are implicit functions of time and $\dot{V}$ is required to be non-positive, the state of the system must be found on sucessively "smaller" $V(x) = \text{const}$ surfaces or must remain stationary. But $\dot{V}(x)$ cannot be zero on any solution except $\underline{x} = 0$, therefore the state of the system cannot remain stationary. Hence, the system trajectory must move toward the origin. The third condition, $V(x) \rightarrow \infty$ as $\| x \| \rightarrow \infty$ insures that all points in the state space will be found on some $V(x) = \text{const}$ surface.

Three other features of the Second Method should be noted. First, the method provides only sufficient conditions for stability; hence if a system does not satisfy the stability theorem, no conclusion may be drawn relative to system stability. Second, the converse of the stability theorem has been proven. Therefore if the system is stable, a V-function must exist. Third, the V-function is <u>not</u> unique, which is one of the most powerful features of the Second Method. No longer is one searching for a single unique solution to the differential equation but rather for one, out of many, V-functions. However because the method provides only sufficient conditions, some V-functions may provide a better answer than others.



$K_1 < K_2 < K_3$
$t_1 < t_2 < t_3$

Surfaces of V(x) = Const

Fig. 1

III. Linearization

One tool which has been used extensively in analyzing nonlinear systems is equivalent linearization. To quote LaSalle and Lefschetz: " ... in determining practical stability, linear approximations are definitely unsatisfactory." Since linear stability is always global, it is impossible to determine from linear approximations the extent of asymptotic stability -- this must be done by examining the nonlinearities. In fact, the method of equivalent linearization is only justified on the basis of Liapunov's stability theorems as indicated below.

The Second Method, however, not only justifies linearization but it makes it possible to determine a finite region of asymptotic stability. If $V(x)$ is chosen to be a positive definite quadratic form, then $V(x)$ satisfies conditions (a) and (c) of the stability theorem. In addition, $V(x)$ can be chosen such that for the linearized system $\dot{V}$ is negative definite. If the nonlinear system is used with the same V-function, there must be a finite region about the origin where $\dot{V}$ is negative, since for small $\| \underline{x} \|$ the linear terms will predominate. Now by selecting the largest $V(\underline{x}) = \text{const}$ surface that will fit into this region, a finite region of stability will be specified. Consider the following method for obtaining such a V-function.

Let each of the terms in the $B(\underline{x})$ matrix be written as a power series about the origin. This is normally not difficult since many of the nonlinearities are already in this form and usually only a small number of the b_{ij}'s are functions of $\underline{x}$. Now write $B(\underline{x})$ as the sum of a constant matrix A and a non-constant matrix $C(\underline{x})$

$$B(\underline{x}) = A + C(\underline{x}) \tag{5}$$

Then equation (2) becomes:

$$\dot{\underline{x}} = A\,\underline{x} + C(\underline{x})\,\underline{x} \tag{6}$$

This is equivalent to linearization about the origin. Now consider only the linearized system:

$$\dot{\underline{x}} = A\,\underline{x} \tag{7}$$

A linear transformation of coordinates, $x = Pz$, may be made such that in terms of the new coordinates

$$\dot{\underline{z}} = P^{-1} A P\underline{z} = D\underline{z} \tag{8}$$

where D is a diagonal matrix containing the eigenvalues as its elements. If the linear system is stable, the real part of each eigenvalue must be negative. Now choose $V(\underline{z})$ as:

$$V(\underline{z}) = \bar{\underline{z}}' \underline{z} \tag{9}$$

where $\bar{\underline{z}}$ represents the conjugate of $\underline{z}$. Then for the linear system, $\dot{V}(\underline{z})$ is given by:

$$\dot{V}(\underline{z}) = \bar{\underline{z}}' (\bar{D}' + D)\, z \tag{10}$$

which must be negative definite. Therefore, a V-function has been found for the linearized system. However, since the $\underline{x}$ state variables are more related to the physical problem, it is necessary to find V in that coordinate system in order to be able to interpret the results. This is a relatively simple job, accomplished by substituting for $\underline{z}$ in equation (9), $\underline{z} = P^{-1}\,\underline{x}$

$$V(x) = \underline{x}' \bar{P}^{-1'} P\,\underline{x} \tag{11}$$

Again for the linear system, $V(x)$ and $\dot{V}(x)$ must be positive and negative definite respectively. The only remaining steps are to compute $\dot{V}(x)$ using the nonlinear system and to find the largest $V(x)$ = const surface which is contained in the region where $\dot{V}(x)$ is negative. A numerical example of this technique is presented below in example No. 1.

Example 1

The model chosen for discussion is one in which negative reactivity is introduced into the reactor as a result of an increase in the temperature of the core. It is assumed that the properties of the reactor and the conditions within the core may be averaged such that the equations adequately describe the overall dynamic behaviour of the system independent of spatial effects.

If the reactor is unreflected and if one group of delayed neutrons is considered, the neutron kinetics equations are written

$$\dot{n}(t) = \frac{1}{\ell}\,\rho(t)\,n(t) - {}^{\beta}/_{\ell}\,n(t) + \lambda c(t) \tag{12}$$

$$\dot{c}(t) = {}^{\beta}/_{\ell}\,n(t) - \lambda c(t) \tag{13}$$

where

n = neutron density (neut/cc)

C = delayed neutron precursor density (nuclei/cc)

β = delayed neutron fraction

λ = delayed neutron precursor decay constant (sec^{-1})

ℓ = generation time (sec)

ρ = reactivity

The linearly temperature-dependent reactivity effect is written

$$\rho(t) = \rho_0 - a' T(t) \tag{14}$$

where

a' = temperature coefficient of reactivity ($^oF^{-1}$)

T = temperature (^oF) at zero power equilibrium

and the relationship between temperature and the neutron density or power level is arrived at by means of an energy balance on the reactor.

$$\dot{T}(t) = Kn(t) - hT(t) \tag{15}$$

The constants K and h represent a conversion factor and a linear heat removal process, respectively. Equation (4) is written under the assumption that the coolant flow is constant.

In order that the origin of the state plane or space represents the equilibrium point of the system, the following transformation of variables is considered:

$$x = \frac{n - no}{no}$$

$$y = \frac{T - To}{To} \tag{16}$$

$$z = \frac{C - Co}{Co}$$

The complete set of equations becomes

$$\dot{x} = - ay (1 + x) - bx + cz \tag{17}$$

$$\dot{y} = dx - cy \tag{18}$$

$$\dot{z} = fx - gz \tag{19}$$

where

$$a = a'To/\ell$$

$$b = c = \beta/\ell$$

$$d = \frac{no}{m\ cp\ To}$$

$e = \dot{m}/m$

$f = g = \lambda$

m = mass of coolant in core

$\dot{m}$ = mass flow of coolant through core

cp = coolant specific heat

no = operating power level in kilowatts

If the delayed neutrons are not considered, equation (8) is eliminated and $b = c = o$. The dynamics are then described by equations (9).

$$\begin{aligned} \dot{x}(t) &= - ay(t) \left[1 + x(t)\right] \\ \dot{y}(t) &= dx(t) - ey(t) \end{aligned} \tag{20}$$

Applying the method of obtaining $V(x)$ described in Section (III), equations (9) are written in the matrix form.

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} o & -a \\ d & -e \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} -a \\ o \end{Bmatrix} x_1 \, x_2$$

where

$x_1 = x(t)$

$x_2 = y(t)$.

If R and $\bar{R}$ are the complex conjugate eigenvalues of the linearized matrix,

$$P = \begin{bmatrix} -a & -a \\ R & \bar{R} \end{bmatrix}$$

and $P^{-1} = \dfrac{1}{a(R - \bar{R})} \begin{bmatrix} \bar{R} & a \\ -R & -a \end{bmatrix}$.

where $R, \bar{R} = \frac{1}{2} e \pm \sqrt{ad - \frac{1}{4} e^2}\ j$.

The operation of equation (11) yields

$$V(x) = \frac{1}{2a^2} \left\{ x_1^2 - \frac{e}{d} x_1 x_2 + \frac{a}{d} x_2^2 \right\} \tag{21}$$

The linearized $\dot{V}(x)$ is

$$\dot{V}_{\ell}(x) = \frac{1}{2a^2 d} \left[- ed\, x_1^2 + e^2 x_1 x_2 - ae\, x_2^2 \right] \tag{22}$$

and the nonlinear term to be added to the cross product $x_1 x_2$ is:

$$\dot{V}_n(x) = \frac{1}{2a^2 d} \left[- ax_1x_2(2dx_1 - ex_2) \right] \tag{23}$$

where $\dot{V}(x) = \dot{V}_{\ell}(x) + \dot{V}_n(x)$.

Two cases are considered for comparison in Figs. 2 and 3.

1. Pressurized water reactor operating at 200 mw. (Fig. 2)

   a = 2000 b = 100

   d = 10 f = 0.1

   e = 0.5

2. Nuclear Rocket (Fig. 3)

   $a = 10^6$ b = 500

   d = 2.0 f = 0.1

   e = 1.5

The largest $V(x)$ which will fit in the region of $\dot{V}(x) > 0$ is shown as a circle in each of the figures. The region is indeed small as is expected, with the region for the rocket example being the smaller. The latter observation is expected physically in light of the fact that the rocket is operating at 1000 times the power as the PWR and thus a given displacement in power for the two cases is fractionally smaller for the rocket.

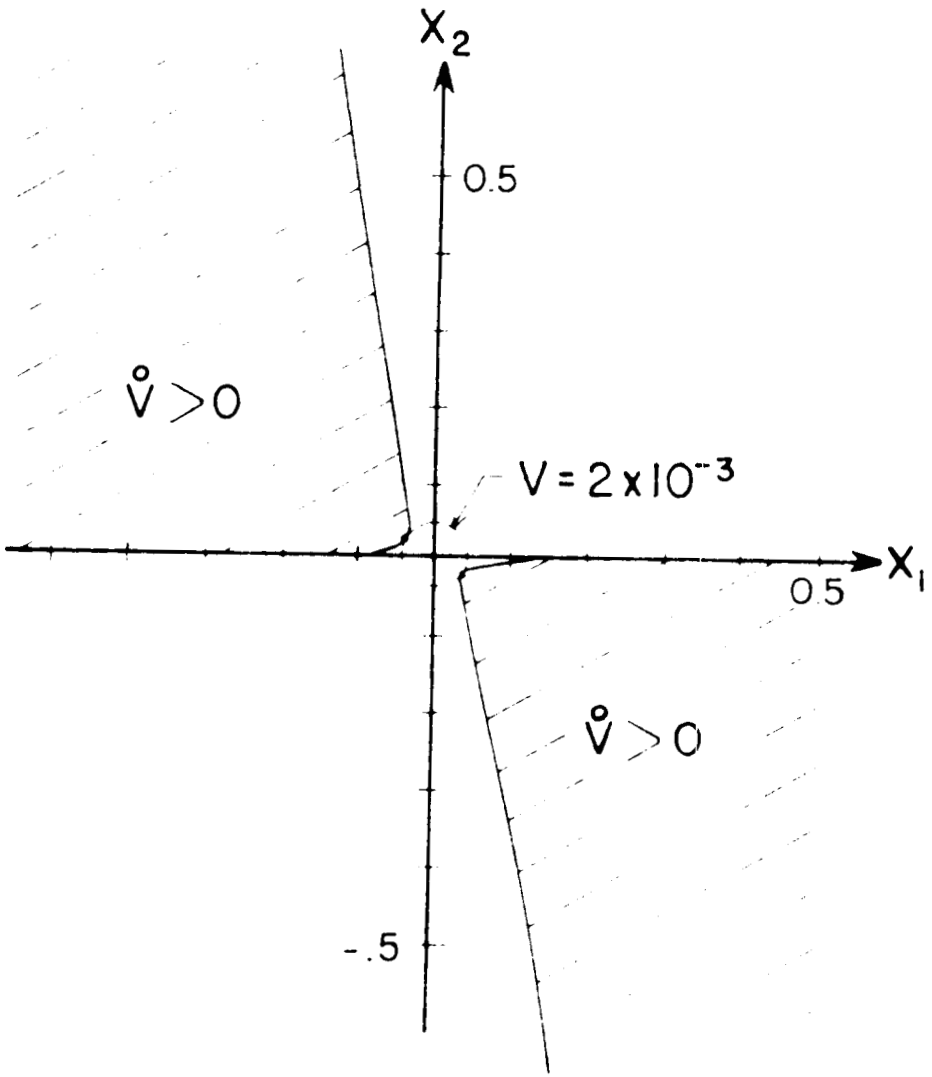


Fig. 2

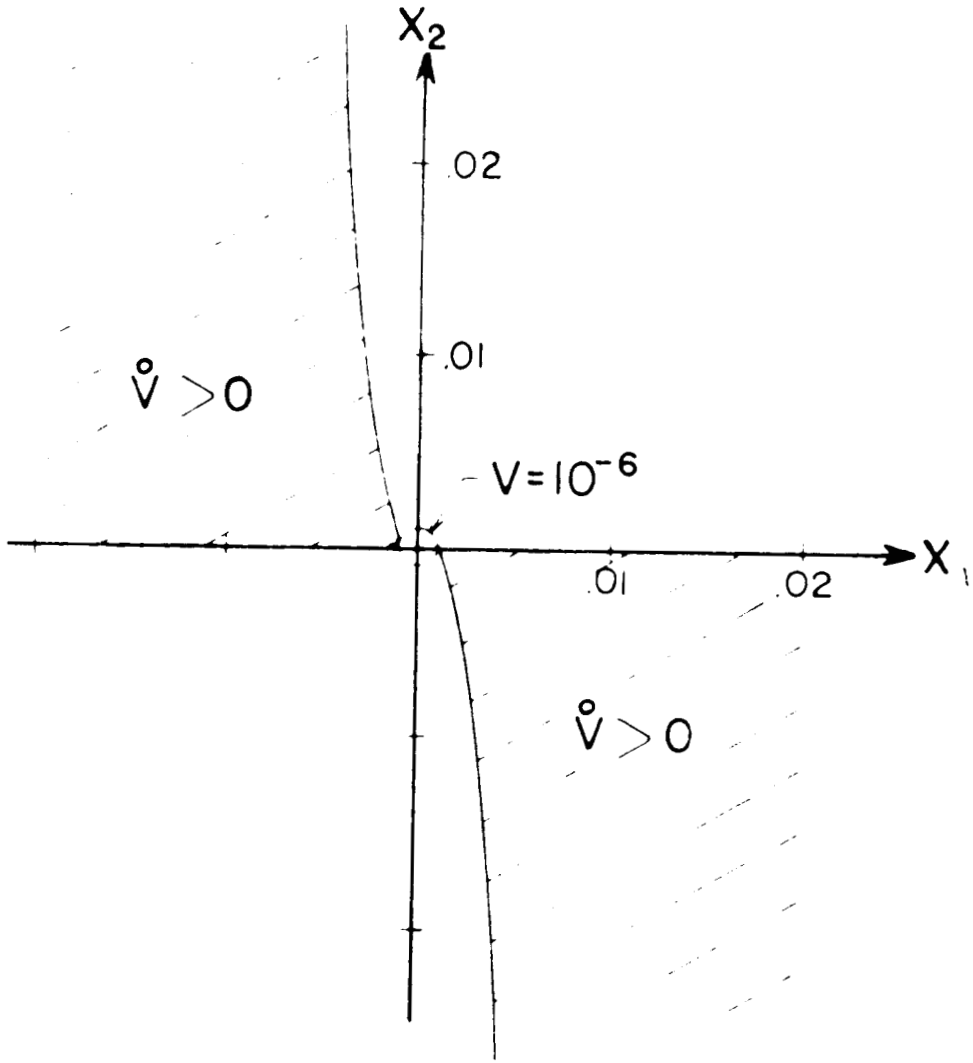


Fig. 3

IV. Lurie-Letov Method

The principal contributions to the nuclear engineering literature in the area of the Second Method have been made by Gyftopoulos, Levin, Nohel, Smets, and Timlake. Each author makes use of a Liapunov function of the Lurie type, in which the conditions for stability are based upon the simultaneous solution of a set of so-called stability equations. This approach is the basis for the book by Letov, available in English. Lurie forms require unconditional stability of the linearized equations.

The meaning of this statement may be clarified by consideration of an nth order system with one single-valued nonlinearity, $f(\underline{x})$. The linear portion of the system is assumed to have distinct non-zero eigenvalues. The form of the given equations is usually arranged in the so-called Lurie's cannonic form, in which the ith equation appears as

$$\dot{y}_i = \lambda_i y_i + f(\underline{x}) \tag{24}$$

This is equivalent to matrix diagionalization. Lurie's method then seeks a set of stability equations or conditions which will be valid for every $f(\underline{x})$. Herein lies the power as well as the weakness of the method. The nonlinearity may not be severly nonlinear at all. In fact, it may be a simple linear gain, the least severe of all nonlinearities. If this is the case, the Lurie conditions must be satisfied for all values of this gain, or for values ranging from zero to infinity. In short, the system must be unconditionally stable.

A third order system with an unbounded gain, as for instance x^3, is represented in block diagram form in Fig. 4. This system is not

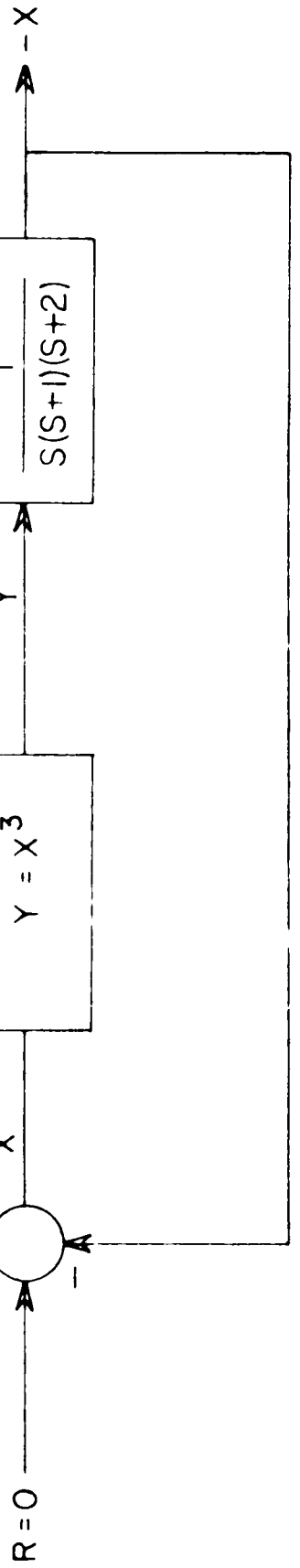


Fig. 4

unconditionally stable, as the root locus of the system crosses into the right half plane for K greater than 6. Hence, none of Lurie's forms could ever be found to prove global asymptotic stability. This is true not just because the nonlinearity was chosen as x^3, but this would be true for any nonlinearity, as the Lurie method considers all nonlinearities jointly.

Rather than assume a Liapunov V-function and derive stability conditions valid for any nonlinearity, an alternate approach might be to attempt to determine one V-function that would fit a particular case. This is the approach that has been actively pursued by control engineers in this country. Contributions have been made toward this end by Donaldson, Ingwerson, Rekasius, Schultz, and Szego. For example, in the third order system considered in Fig. 2 above, a V-function suitable for proving asymptotic stability for an initial value of x less than or equal to $\sqrt[3]{6}$, the value beyond which Routh Hurwitz says even the linear system is not stable is

$$V = \frac{3}{4} x_1^4 + 2x_1^2 + 6x_1x_2 + \frac{11}{2} x_2^2 + 2x_1x_3 + 3x_2x_3 + x_3^2 \qquad (25)$$

when

$$x_1 = \sqrt[3]{6} \quad \text{and} \quad x_2 = x_3 = 0$$

$$V = 3/4 \sqrt[3]{6} + 2(6)^{2/3} = V_1 \qquad (26)$$

The value of V_1 represents the largest closed surface within which the system may start and be guaranteed to return to the origin. In this case, V is positive definite in the entire space but $\dot{V}$ ceases to be negative semi-definite at $x_1 = \sqrt[3]{6}$.

It is important to emphasize that the V-function stated above was determined, or rather generated, to fit this specific problem. No stability

equations need be satisfied. All that is necessary is that V and $\dot{V}$ meet the conditions of the Liapunov theorem. The method advocated by the authors is the Variable Gradient method of generating Liapunov functions. This method is presented in the following section.

It is further shown that the Lurie-Letov approach, used by Gyftopoulos, Levin, Nohel, Smets, and Timlake, in nuclear applications is unnecessarily restrictive and bound to fail to prove stability in a large number of cases when stability actually does exist. In order to overcome the obvious short-comings of the Lurie-Letov technique, a means is presented of generating a particular V-function to fit the problem at hand. This approach, called the Variable Gradient Method, overcomes the deficiencies of the Lurie-Letov approach and requires no particular state variable representation, such as the Lurie canonic form. Special emphasis is placed on the use of the "natural" state variables, directly related to the physical system. In this way direct physical interpretation and intuition may be applied to the problem.

Examples are presented to illustrate the mechanics of the method and the advantages of the use of the "natural" state variables. Included are applications to reactor dynamics with or without delayed neutron effects and with temperature dependent reactivity effects which are of practical importance.

V. Variable Gradient Method

As the name implies, the variable gradient method of generating V-functions is based on the assumption of a general vector function ∇V. The gradient of V is a particularly interesting function since both V and $\dot{V}$ can be determined from this vector.

The use of ∇V for determining $\dot{V}$ was mentioned previously in the discussion of the Second Method. The result is repeated here for reference.

$$\dot{V} = \nabla V' \dot{x} . \tag{27}$$

V is obtained as a line integral of ∇V as

$$V = \int_{o}^{\underline{x}} \nabla V' \cdot d\underline{x} \tag{28}$$

The upper limit here is not meant to imply that V is a vector, but rather that the integral is a line integral to an arbitrary point in the phase space located at $(x_1, x_2, \ldots, x_n)$. As shown below, this integral can be made independent of the path fof integration, the simplest of which is indicated by the expanded form of equation 28.

$$V = \int_{o}^{x_1, (x_2=x_3 = \ldots = x_n = 0)} \nabla V_1 \, dx_1 + \int_{o}^{x_2 (x_1=x_1,\ x_3=x_4=\ldots=x_n = 0)} \nabla V_2 \, dx_2$$

$$+ \ \ldots \ + \int_{o}^{x_n (x_1=x_1,\ \ldots,\ x_{n-1} = x_{n-1})} \nabla Vn \, dxn \quad .$$

where the component of the vector ∇V in the x_i direction is $\nabla V_i = \dfrac{\partial V}{\partial x_i}$.

Standard texts on vector calculus show that, for a scalar function V to be obtained uniquely from a line integral of a vector function ∇V, the matrix $\bar{\Phi}$ formed by $\partial \nabla V_i / \partial x$; must be symmetrical; that is

$$\underline{\Phi} = \begin{pmatrix} \frac{\partial \nabla V_1}{x_1} & \frac{\partial \nabla V_2}{x_1} & \cdots & \frac{\partial \nabla V_n}{x_1} \\ \frac{\partial \nabla V_1}{x_2} & \cdots & & \vdots \\ \vdots & & & \\ \frac{\partial \nabla V_1}{x_n} & \cdots & & \frac{\partial \nabla V_n}{x_n} \end{pmatrix} \tag{29}$$

must be a symmetrical matrix. Thus, in the third-order case, for example, the requirement on states that

$$\frac{\nabla V_2}{x_1} = \frac{\nabla V_1}{x_2}, \quad \frac{\nabla V_3}{x_1} = \frac{\nabla V_1}{x_3}$$

and

$$\frac{\nabla V_3}{x_2} = \frac{\nabla V_2}{x_3}$$

These are simply the usual curl equations, which specify that $\nabla \mathbf{x} \nabla V = 0$ for V to be uniquely obtained by a line integral of ∇V. The condition of the matrix $\underline{\Phi}$ is thus a generalized curl requirement for the n-dimensional case.

The problem of determining a V-function which satisfies Liapunov's theorem is then transformed into the problem of finding a ∇V such that the n-dimensional curl of $\nabla V = 0$. Further, the V and $\dot{V}$ determined from ∇V must be sufficient to prove stability; that is, they must satisfy Liapunov's theorem. On the surface, it appears as though the problem is being made more difficult, as two new functions ∇V and $\underline{\Phi}$ have been introduced. Actually, the opposite is true. The curl equations are the device that permits a solution of the stability problem starting with ∇V.

In attempting to generate V-functions for a system with non-linearities expressed in polynomial form, Szego assumes the coefficients of the usual quadratic form are not fixed, but are functions of the state variables. The same assumption is made here concerning ∇V. As a first step in the method, ∇V is set equal to the product of an arbitrary matrix and the state vector $\underline{x}$

$$\nabla V = Q(x)\ \underline{x} \tag{30}$$

where

$$Q(\underline{x}) = \begin{pmatrix} q_{11}(\underline{x}) & q_{12}(\underline{x}) & \cdots & q_{1n}(\underline{x}) \\ q_{21}(\underline{x}) & \cdots & & \\ \vdots & & & \\ \vdots & \cdots & & \\ q_{n1}(\underline{x}) & \cdots & & q_{nn}(\underline{x}) \end{pmatrix} \tag{31}$$

Substituting into equation 27 we have for $\dot{V}$

$$\dot{V} = \underline{x}' Q' \dot{\underline{x}} = \underline{x}' Q' B\ \underline{x}$$

The i^{th} components of ∇V is given by

$$\nabla V_i = \frac{V}{x_i} = q_{i1}(\underline{x})x_1 + q_{i2}(\underline{x})x_2 + \cdots q_{in}(\underline{x})x_n$$

The $q_{ii}(\underline{x})$ must be chosen positive to insure that V has at least a chance of being positive definite. The remaining $q_{ij}(x)$ are left as completely undetermined quantities. Written in this general form, ∇V appears quite formidable. However, several of the q_{ij} often turn out to be constants, including zero, or they are obvious from constraints on V imposed by the investigator, or they are determined from the curl equations. Thus step 1 of the method is accomplished. The complete procedure is outlined herewith.

1. Assume ∇V is of the form shown in equation 30.
2. From ∇V, determine $\dot{V}$.
3. Constrain $\dot{V}$ to be at least semidefinite.
4. Use the $n(n-1)/2$ curl equations implied by the statement that $\underline{\Phi}$ must be symmetric to determine the remaining unknown coefficients in ∇V.
5. Recheck $\dot{V}$, because the addition of terms required as a result of step 4 may alter $\dot{V}$.
6. Determine V by equation 28, and check for the region of closedness.

In practice, it is often necessary to complete part of step 4 before it is possible to finish step 3 completely. Two examples will serve to illustrate the procedure.

The variable gradient method is applied first to equations 17 and 18, the second order case. $\dot{V}(x)$ is given by the product

$$\dot{V} = x' \begin{bmatrix} -a\alpha_{11}(1 + x_1) \\ -e\alpha_{21} \\ -a\alpha_{12}(1 + x_1) \\ d\alpha_{22} \quad -e\alpha_{22} \end{bmatrix}$$

In order to insure that $\dot{V}(x)$ is at least negative semi-definite, the following steps are taken:

1. $\alpha_{21} = 0$
2. Since α_{21} = a constant, $\alpha_{12} = \alpha_{21} = 0$
3. $\alpha_{22} = K > 0$
4. The cross product term must be zero, so $\alpha_{11} = \frac{K^{d/a}}{1+x_1} = \frac{K_1}{1+x_1}$

The corresponding Liapunov functions are

$$V(x) = K_1 \left(\left[x_1 - \ln (1 + x_1) \right] + \frac{a}{2d} x_2^2 \right)$$

$$\dot{V}(x) = -K_1 \frac{a}{d} e\, x_2^2$$

Assuming that $K_1 = 1$, V(x) = Constant is plotted in Fig. 5, with the x_2 axis scaled up by a factor of $(a/2d)^{\frac{1}{2}}$. The dashed curves represent the V(x) arrived at by assuming the general quadratic form discussed previously under the linearization method. (It is recalled that the largest V(x) for the linearized case is $2 \times 10^{3}(10^{-6}$ for the rocket). This represents an extremely small region in Fig. 5.) It is noted that the V(x) obtained from the variable gradient always lies within the region $x_1 > -1$. These are the only values x_1 can assume physically, since for any given operating power, $x_1 = -1$ when the power level goes to zero.

The addition of equation 29, and the delayed neutron effects in equation 27, leads to the third order problem. If $z = x_3$,

$$\dot{V} = x^{1} \begin{bmatrix} -b\alpha_{11} + d\alpha_{21} + f\alpha_{31} & -a\alpha_{11}(1 + x_1) - e\alpha_{21} & c\alpha_{11} - g\alpha_{31} \\ -b\alpha_{12} + d\alpha_{22} + f\alpha_{32} & -a\alpha_{12}(1 + x_1) - e\alpha_{22} & c\alpha_{12} - g\alpha_{32} \\ -b\alpha_{13} + d\alpha_{23} + f\alpha_{33} & -a\alpha_{13}(1 + x_1) - e\alpha_{23} & c\alpha_{13} - g\alpha_{33} \end{bmatrix}$$

The constraint of $\dot{V}(x)$ is accomplished in this case as follows:

1. Let $\alpha_{13} = \alpha_{31} = 0$
2. Set $x_2 x_3$ coefficient = 0
3. Set $x_1 x_2$ coefficient = 0
4. Set the constant part of $x_1 x_3$ coefficient = 0
5. From the experience gained in the second order problem, let $\alpha_{11} = \frac{K_1}{1+x_1}$

The results of steps 2, 3, and 4 are

2. $\alpha_{23} = (\frac{b}{e+f})\ \alpha_{12}$

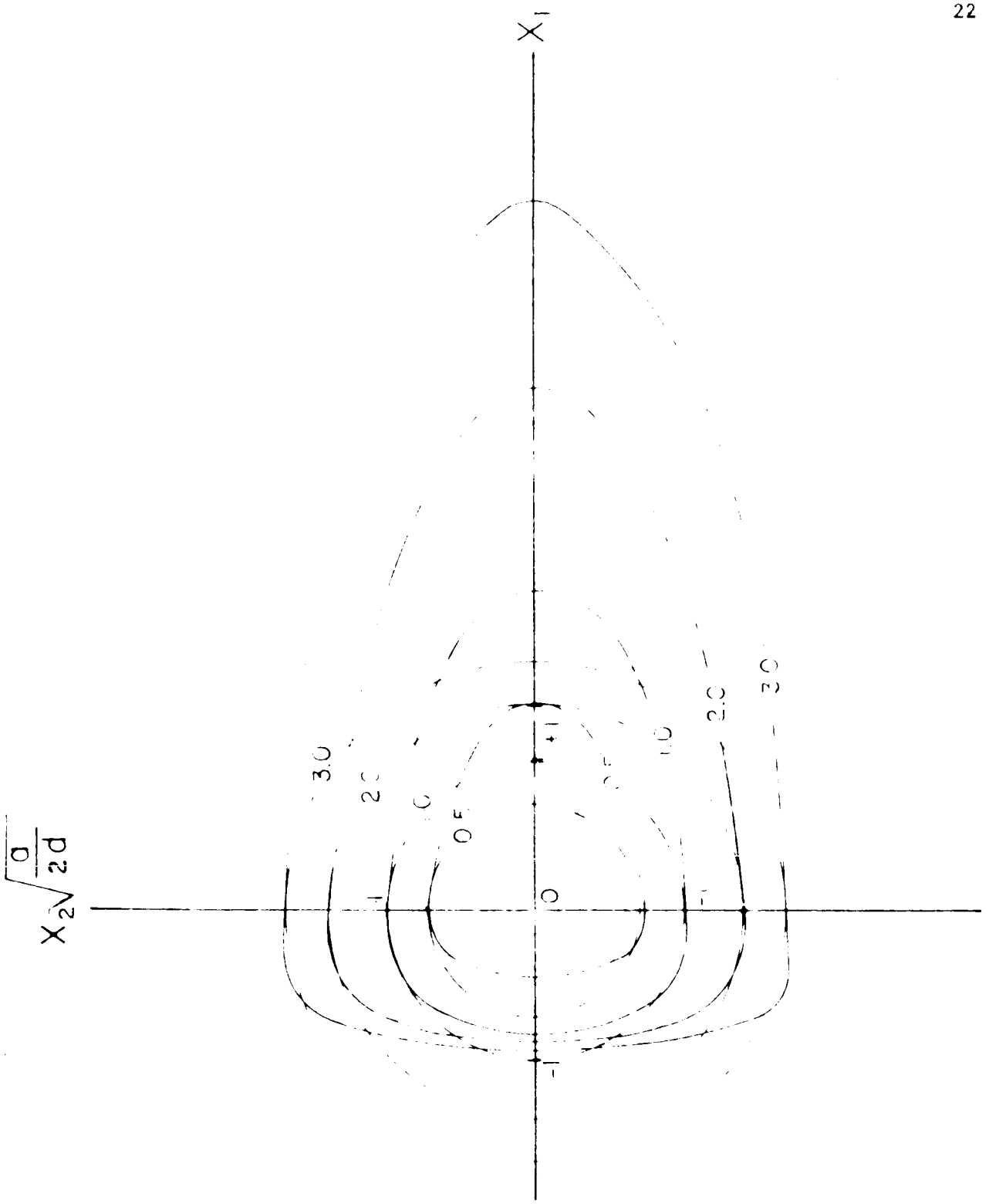


Fig. 5

3. $\alpha_{22} = \frac{1}{d}\left[aK_1 + \alpha_{12}\left(e + b - \frac{bf}{e+f}\right)\right]$

4. $\alpha_{33} = -\frac{bd}{f(e+f)}\alpha_{12}$

It is seen that α_{33} must be a positive number, so let $\alpha_{12} = \alpha_{21} = -K_2$

The results are:

$$\dot{V} = -\left[\frac{K_1 b}{1+x_1} + K_2 d\right] x_1^2 - \left[K_1 \frac{a}{d} e - K_2 a(1+x_1) + K_2 \frac{e^2}{d}\left(\frac{e+b+f}{e+f}\right)\right] x_2^2$$

$$-K_2 \frac{bd}{e+f} x_3^2 + \frac{K_1 b}{1+x_1} x_1 x_3$$

$$V = K_1\left[x_1 - \ln(1+x_1)\right] + \left[K_1 \frac{a}{2d} - K_2 \frac{e}{2d}\left(\frac{e+b+f}{e+f}\right)\right] x_2^2$$

$$+ K_2 \frac{bd}{f(e+f)} x_3^2 - K_2 \frac{b}{e+f} x_2 x_3$$

Inserting typical numbers for a pressurized water reactor, and again setting $K_1 = 1$,

$$\dot{V} = 1\left[\frac{100}{1+x_1} + 10K_2\right] x_1^2 - \left[100 - 2000K_2(1+x_1)\right] x_2^2 - 1700K_2 x_3^2$$

$$+ \frac{100}{1+x_1} x_1 x_3$$

$$V = \left[x_1 - \ln(1+x_1)\right] + \left[100 - 5K_2\right] x_2^2 + 1.7 \times 10^5 K_2 x_3^2$$

$$- 1.7 \times 10^2 K_2 x_2 x_3$$

For small values of x_1, the coefficient of x_2^2 becomes positive if $K_2 < 1/20$. This may be a guide in selecting K_2, although a larger value does not necessarily indicate that $\dot{V}(x)$ is positive for regions of interest in the state space. The sign definiteness of the second order terms of $V(x)$ is preserved if $K_2 < 20$.

The effect of the addition of delayed neutrons to the system is seen most clearly in the x_1^2 term of $\dot{V}(x)$, where the delayed neutron fraction appears as a relatively large negative coefficient.

Although the visualization of the effect of varying K_2 is hindered by the presence of three dimensions, it is helpful to examine the regions in each of the three planes separately. Graphical illustrations of this procedure are shown in Figs. 6a, b. It is seen that the size of the region in which $\dot{V}$ is negative semi-definite may be increased in one place while it decreases in another.

As an example of choosing a V(x), consider the case where $K_2 = \frac{1}{100}$. The largest V(x) which will fit in the region $\dot{V} \leq 0$ is V(x) = 17.

VI. Summary

The advantages of the Variable Gradient method are evident from the examples. In the second order case, the linearization technique leads to a region of stability which is extremely small whereas the variable gradient leads to a conclusion of stability for all physically real values of the variables. It is possible that the smaller region may be adequate for cases in which the system conditions would never be expected to change. However, for a system which would encounter a wide range of conditions and large incremental power demands, such as a nuclear rocket, it is necessary to be able to conclude stability for as large a region as possible.

The desirability of employing the natural system variables is seen in the ease with which the Variable Gradient method is applied with the equations in their original form. It is possible to combine the equations in phase variable representation. This requires unnecessary manipulation and leads to combersome nonlinear terms. Also, some physical intuition may

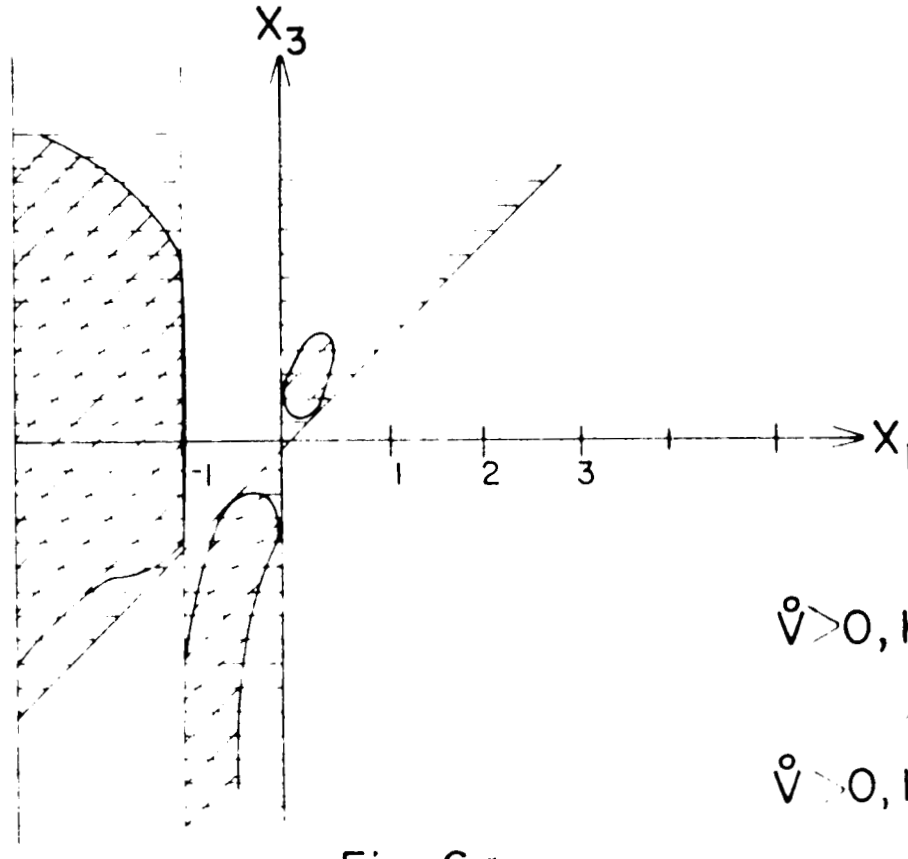


Fig. 6a

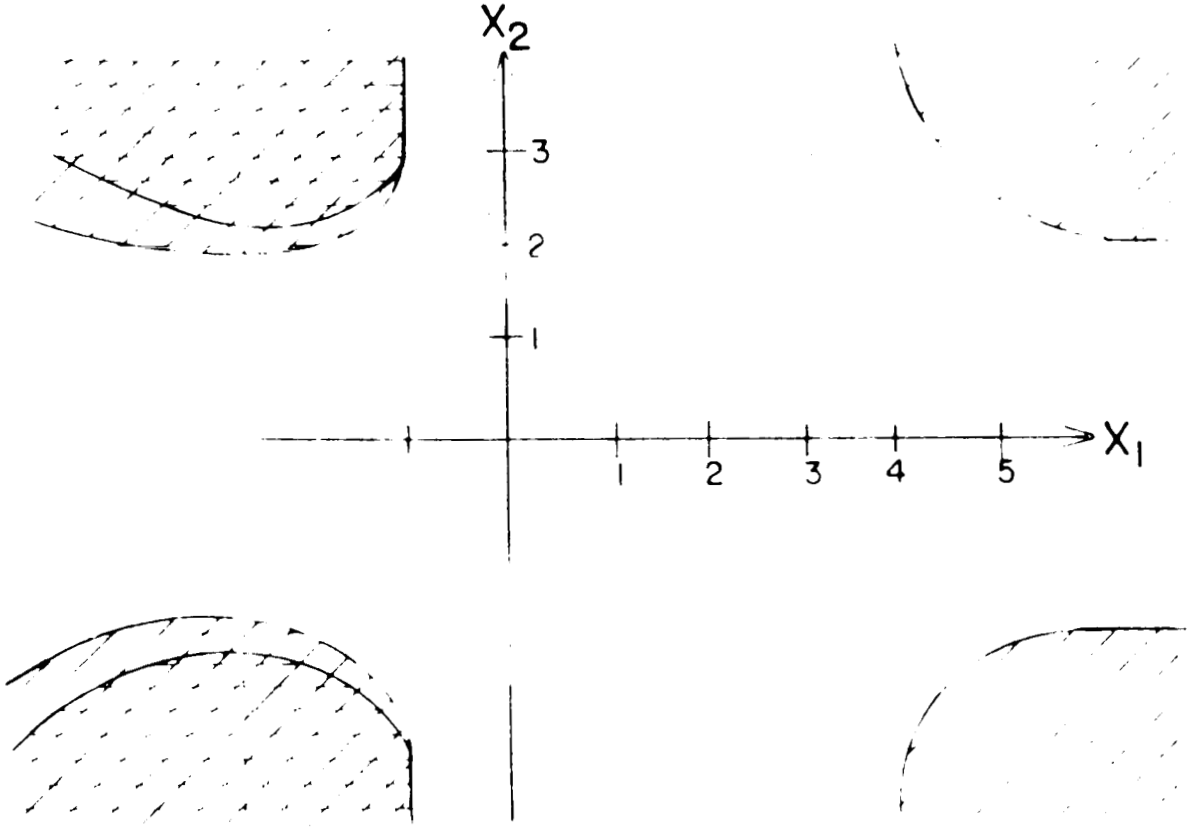


Fig. 6b

be used in the analysis because permissable limits or the temperature and power level are known.

Regardless of the technique used in applying the Second Method, higher order problems become somewhat difficult to approach. The Variable Gradient method reduces the labor somewhat by eliminating preliminary manipulations. For lower order problems it is noted that nonlinearities more complicated than the one considered in this report may be handled by the Variable Gradient method without appreciably increasing the difficulty of the problem. A general problem in which the reactivity is a nonlinear function of temperature or a function of more than one variable such as in a nuclear rocket, may be worked to include an extremely wide range of operating conditions.

VII. Research to be Conducted During the Next Report Period

Experience gained in the application of the Second Method of Liapunov to basic examples leads to a consideration of more difficult practical problems. In particular, for the analysis of nuclear rocket symptoms, the following three areas merit investigation.

A. Time Varying Parameter Problems

In addition to reactivity effects inherent in a reactor system, it is important that the externally-regulated reactivity inputs be considered. This is particularly true in the stability analysis of a nuclear rocket system in which the power level would be adjusted over a wide range by use of a programmed reactivity insertion.

Physically, reactivity inputs are looked upon as system driving functions; for purposes of analysis, however, this input appears as a time-varying parameter in the nonlinear kinetics equation.

The most promising approach in applying Liapunov's Second Method to this class of problems is to introduce either time or a function of time as another state variable. This necessarily increases the order of the problem by one, but other methods rely on the supposition that the parameters change very slowly with time. This is not valid for the case of rapidly varying reactivity inputs.

The use of bounded inputs leads to a discussion of system boundedness rather than asymptotic stability. The solution is bounded if there exists a closed region about the equilibrium point outside of which dV/dt is negative definite. In a reactor this corresponds to a bounded increase in the power level in response to a bounded reactivity input such as a step function.

B. Coupled Core Reactors

This study of arrays of independently subcritical cores, the net result of which is a critical configuration, is complicated by the existence of time delay terms in the neutron kinetics equations. This delayed source term arises from the interactions between the individual cores. The stability problem for systems with delay should be treated most generally by the application of Liapunov's Direct Method.

In addition to an investigation of asymptotic stability, the effect on each core of a reactivity input to one or several cores is important. The methods found to be useful in the analysis of time varying parameter problems should be directly applicable in this case.

Although the coupled core stability analysis would deal with a high order system, there is considerable similarity and symmetry among the equations, thus the solutions should not be as formidable as for other other

high order problems. This problem has additional fundamental importance since a further study could lead to an understanding of the spatially dependent dynamics of reactor cores.

C. Suboptimal Control of Nonlinear Systems

In recent years there has been an ever increasing gap between practice and theory in the area of optimal control. All of the techniques presently employed for designing optimal systems involve excessive amounts of computational labor particularly in the nonlinear case. The resulting systems are generally open loop in nature and hence very sensitive to parameter variations and external disturbance. Also, in general, little or no physical intuition is gained from these solutions.

One approach which has been suggested for alleviating these problems is suboptimal systems, systems which do not provide the optimum in performance but which are more easily realized in a practical closed loop form. Since the performance index used is arbitrary in many cases, the need for complicated systems which minimize the performance index may be highly questionable. It is proposed that an investigation of suboptimal control be undertaken using as a basic tool the Liapunov function. With this approach it is necessary that the autonomous system be (asymptotically) stable -- this would not appear to be overly restrictive. In the course of the study, it will undoubtedly become necessary to make several other restrictions of the class of system to be investigated or the performance indices employed.

A suboptimal design technique is ideally suited to problems, such as the nuclear rocket control system, in which the describing equations are highly approximate. The use of a complicated design procedure to find

an optimum control in this case is questionable particularly since the answer will generally be in the form of open loop control. If it were possible to design a practical closed loop system which performs in an approximately optimum fashion this would certainly be the more desirable engineering solution.

PART II

Optimization Studies in Nuclear Reactor Dynamics

Optimum Reactor Shutdown Program for Minimum Xenon Buildup

I. Introduction

After shutdown of a high-flux thermal reactor, the xenon concentration will increase for many hours. Since xenon formation is the result of the radioactive decay of iodine, the maximum xenon concentration depends also on the iodine concentration at the time of shutdown. The maximum xenon concentration increases rapidly with flux level. For a flux level of 10^{14} neutrons per square centimeter per second the maximum xenon concentration after shutdown is 4 times the equilibrium xenon concentration at time of shutdown. The maximum xenon concentration corresponding to a flux level of 10^{15} neutrons per square centimeter per second is 50 times greater than the equilibrium value. In order to start the reactor any time after shutdown, sufficient excess reactivity must be present to override the xenon poisoning. The amount of excess reactivity required to override the xenon poisoning can be minimized by determining an optimum reactor shutdown program. The development of optimum reactor shutdown programs have been investigated by Ash, Bellman and Kalaba using dynamic programming.[1] Due to the size of computer necessary to solve this problem no solution was given. Fresdall and Babb[2] investigated the effect of a simple-pre-selected shutdown program, such as exponential or linear functions, in the reduction of the maximum xenon concentration. These programs are in no sense optimal. The approach to the solution of the problem presented here is based on Pontryagins Maximum Principle.[3]

Pontryagins Maximum Principle is used to determine an optimum reactor shutdown program that will minimize the minimum value of xenon

concentration. Optimum shutdown programs have been determined for flux levels ranging between 10^{14} n/cm^2-sec to 10^{16} n/cm^2-sec and for control times up to 7 hours. Results show that considerable reduction in the peak xenon concentration can be accomplished by relatively simple shutdown programs.

II. Formulation of the Problem

In the model used here, it is assumed that the state of the reactor can be described by the neutron flux ϕ, the iodine concentration I, and the xenon concentration X. The equations describing the xenon concentration as a function of time are:

$$\dot{I} = \gamma_1 \Sigma_f \phi - \lambda_1 I - \sigma_1 \phi I \qquad (1)$$

$$\dot{X} = \gamma_2 \Sigma_f \phi + \lambda_1 I - \lambda_2 X - \sigma_2 \phi X \qquad (2)$$

with the initial condition

$$\phi(o) = \phi_o \qquad X(o) = X_o \qquad I(o) = I_o \qquad (3)$$

The symbols used in the above equations are defined below:

γ_1 = fission yield of iodine

γ_2 = fission yield of xenon

λ = decay constant

Σ_f = macroscopic fission cross section

σ = microscopic absorption cross section

The last term in equation (1) is neglected since its contribution is negligible for flux levels less than 10^{16} n/cm^2-sec. Solving equations (1) and (2) for $\phi = 0$, the iodine and xenon concentration as a function of time after shutdown is given by equations (4) and (5) respectively,

$$I = I_b e^{-\lambda_1 t} \tag{4}$$

$$X = \frac{\lambda_1}{\lambda_2 - \lambda_1} I_b (e^{-\lambda_1 t} - e^{-\lambda_2 t}) + X_b e^{-\lambda_2 t} \tag{5}$$

with the initial conditions

$$I(b) = I_b \qquad X(b) = X_b \tag{6}$$

In equations (4) and (5) the subscript b denotes the iodine and xenon concentration at the time of shutdown. Taking the derivative of equation (5) and equating to zero the time corresponding to the occurrence of maximum xenon concentration can be obtained

$$\frac{dX}{dt} = \frac{\lambda_1 I_b}{\lambda_2 - \lambda_1} (\lambda_2 e^{-\lambda_2 t} - \lambda_1 e^{-\lambda_1 t}) - \lambda_2 X_b e^{-\lambda_2 t} = 0 \tag{7}$$

$$t = \frac{1}{\lambda_1 - \lambda_2} \ln \frac{\lambda_1 \lambda_2 (I_b + X_b) - \lambda_2^2 X_b}{\lambda_1^2 I_b} \tag{8}$$

Substituting equation (8) into equation (5), the maximum value of the xenon concentration after shutdown, in terms of the iodine and xenon concentration at the time of shutdown, is given by

$$X_{max} = (I_b + AX_b) (B + AB \frac{X_b}{I_b})^C \tag{9}$$

$$A = 1 - \frac{\lambda_2}{\lambda_1} \qquad B = \frac{\lambda_2}{\lambda_1} \qquad C = \frac{\lambda_2}{\lambda_1 - \lambda_2} \tag{10}$$

The problem is to determine a flux program for the control time interval $0 < t < b$ such that the maximum value of xenon concentration given by equation (9) will be a minimum. From physical considerations the following constraints are placed on the system.

$$\phi_{max} \geq \phi(t) \geq 0 \quad I(t) \geq 0 \quad X(t) \geq 0 \tag{11}$$

III. Application of the Maximum Principle

In applying the maximum principle, the following state variables are defined,

$$x_1 = I \qquad x_2 = X \tag{12}$$

and the standard notation $\mu = \phi$ for the control variable is used. To minimize X_{max} the performance index S is

$$S = X_{max} = (x_{1b} + Ax_{2b})\ (B + AB\ \frac{x_{2b}}{x_{1b}})^C \tag{13}$$

The new state variable x_o is introduced.

$$x_o = (x_1 + Ax_2)\ (B + AB\ \frac{x_2}{x_1})^C \tag{14}$$

The problem reduces to one where the final value of the state variable x_o has to be minimized. The time derivitive of x_o is

$$\dot{x}_o = \mu \Big[(\gamma_1\bar{\Sigma}_f + A\gamma_2\bar{\Sigma}_f - A\sigma_2x_2)(B + AB\ \frac{x_2}{x_1})^C + (\gamma_2\Sigma_f - \sigma_2x_2 - \gamma_1\bar{\Sigma}_f\ \frac{x_2}{x_1} + A\gamma_2\bar{\Sigma}_f\ \frac{x_2}{x_1} - A\sigma_2\ \frac{x_2^2}{x_1} - A\gamma_1\bar{\Sigma}_f\ \frac{x_2^2}{x_1^2})\ x$$

$$x\ (B + AB\ \frac{x_2}{x_1})^{C-1}\ ABC\Big] + (\lambda_1(A-1)x_1 - A\lambda_2x_2)\ (B +$$

$$+ AB\ \frac{x_2}{x_1})^C + ABC\ (B + AB\ \frac{x_2}{x_1})^{C-1}\ \Big[\lambda_1x_1 - \lambda_2x_2 - \lambda_1x_2 +$$

$$+ A(\lambda_1x_2 - \lambda_2\ \frac{x_2^2}{x_1} - \lambda_1\ \frac{x_2^2}{x_1})\Big] \tag{15}$$

The Hamiltonian of the system is defined by

$$H = \sum_{o}^{2} \psi_s\ \dot{x}_s \tag{16}$$

which becomes

$$H = \psi_o \overset{o}{x}_o + \psi_1(\gamma_1 \Sigma_f \mu - \lambda_1 x_1) + \psi_2(\gamma_2 \Sigma_f \mu + \lambda_1 x_1 - \lambda_2 x_2 - \sigma_2 x_2 \mu) \tag{17}$$

where x_o is given by equation (14). The maximum principle states that for a control, μ, to be optimal, X_{max} equal to a minimum, the Hamiltonian of the system must be a maximum with respect to μ for all values of t in the control time interval. For the case in point, the Hamiltonian is a linear function of μ, therefore the optimal control will be an on-off type control operating on the boundaries of the control region. That is

$$\text{for } \frac{\partial H}{\partial \mu} > 0 \qquad \mu = \phi = \phi_{max}$$

$$\text{for } \frac{\partial H}{\partial \mu} < 0 \qquad \mu = \phi = 0 \tag{18}$$

and at every time where

$$\frac{\partial H}{\partial \mu} = \psi_o \left[(\gamma_1 \bar{\Sigma}_f - \sigma_1 x_1 + A\gamma_2 \bar{\Sigma}_f - A\sigma_2 x_2)(B + AB \frac{x_2}{x_1})^C + \right.$$

$$+ ABC \left(\frac{\gamma_2 \bar{\Sigma}_f}{x_1} - \sigma_2 \frac{x_2}{x_1} - \gamma_1 \bar{\Sigma}_f \frac{x_2}{x_1^2} + \sigma_1 \frac{x_2}{x_1}\right)(x_1 + Ax_2)(B +$$

$$\left. + AB \frac{x_2}{x_1})^{C-1}\right] + (\gamma_1 \bar{\Sigma}_f - \sigma_1 x_1)\ \psi_1 + (\gamma_2 \bar{\Sigma}_f - \sigma_2 x_2)\ \psi_2 = 0 \tag{19}$$

the flux will switch from ϕ_{max} to 0 or vice-versa. The next logical step is to determine the optimal number of switchings. This is normally done by calculating the number of times $\frac{\partial H}{\partial \mu}$ changes sign in the control interval. For the case here, the limitations on the state functions are given by equation (11). The auxiliary functions are defined as

$$\overset{o}{\psi}_i = -\sum_{s=0}^{2} \psi_s \frac{\partial \overset{o}{x}_s}{\partial x_i} \tag{20}$$

with the boundary conditions

$$\psi_o(b) = -1 \qquad \psi_1(b) = 0 \qquad \psi_2(b) = 0 \tag{21}$$

These functions are not limited in any other sense. It is not possible from this information to determine the optimum number of switchings. However, an alternate approach can be used. By assuming a given number of switchings it is possible to calculate $\dot{X}$ and X from equations (1) and (2) for the time interval $0 < t < b$. Solutions to these equations will have the switching times $t_{s1}, t_{s2}, \ldots t_{sn}$, as parameters. Knowing $\dot{X}$ and X or x_1 and x_2 and the boundary conditions at $t = b$, the auxiliary functions ψ_1 and ψ_2 can be determined for the control interval $0 < t < b$ from equations (20) and (21) starting with the last sub-interval $t_{sn} < t < b$. These equations will have the same parameters t_{si}. It is now possible to write one equation in the form of equation (19) for every switching point. There will therefore be n algebraic equations with n unknowns corresponding to the switching times.

Another method to solve this problem is to calculate X_{max} from equation (1) for a control function with an assumed number of switchings. By trial and error a combination of switching times can be determined that will yield the smallest xenon peak. Using this technique the optimum number of switchings is also determined. Since this technique is amenable to digital computer calculations it is the approach that was used.

IV. Results

Using the procedure outlined in the previous section, optimum shutdown programs were determined for a reactor having the following values for the constants:

$\gamma_1 = 0.056$ $\lambda_1 = 2.9 \cdot 10^{-5}$ sec^{-1} $\Sigma_f = 10^{-3}$cm^{-1}

$\gamma_2 = 0.003$ $\lambda_2 = 2.1 \cdot 10^{-5}$ sec^{-1} $\sigma_2 = 3.5 \cdot 10^{-18}$cm^2

Figure 1 shows the reduction in the xenon peak as a function of the control time parameter b. The maximum xenon concentration is normalized to the maximum xenon concentration occurring after instantaneous shutdown. Values of the maximum xenon concentration occurring after instantaneous shutdown are shown in Fig. 2 as a function of the flux level. In Fig. 1 there are two distinct regions corresponding to one-pulse and two-pulse control.

One-pulse control is optimal for control times less than four or five hours, depending upon flux level. An example of this type of control is given in Fig. 3. In this figure, the xenon concentration, as a function of time, for instantaneous shutdown is compared with a one-pulse optimum control shutdown program. The pulse width and termination time, for the flux level indicated, are shown in the upper part of the figure. A program of this type can be characterized by one variable, such as the pulse width. Figure 4 gives the optimal pulse width corresponding to given control times and flux levels.

With an increasing control interval the value of the xenon concentration occurring just prior to the control pulse increases rapidly and for b = 4.5 hours it reaches the value of the xenon peak after shutdown. Beyond this point the maximum xenon concentration occurs before, rather than after b = 5. The problem statement given above must therefore be modified. It is necessary to require the maximum value of the xenon concentration to be a minimum after t = 0 rather than after t = b. To meet this requirement a

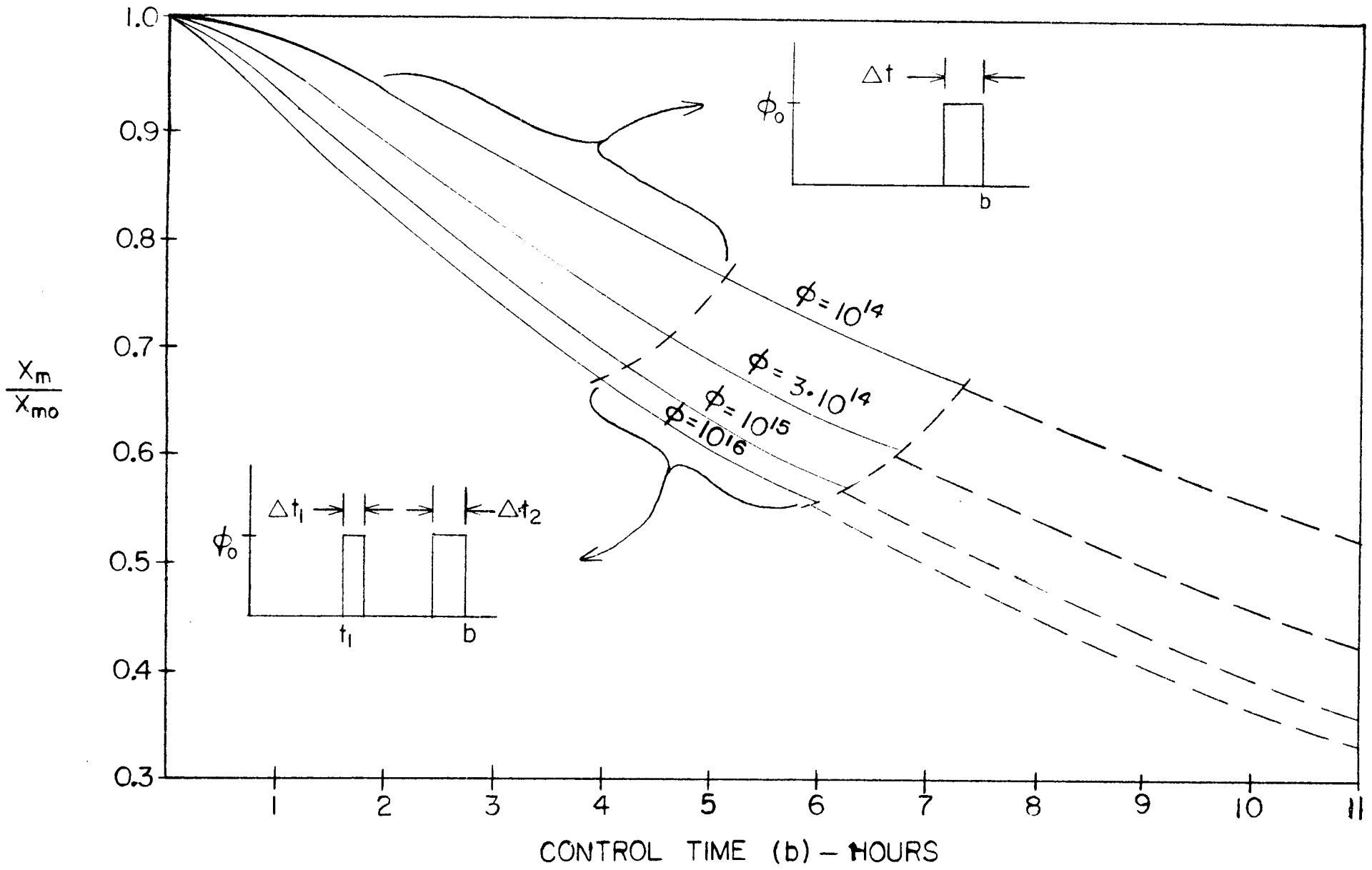


Fig. 1. Reduction in Xenon Buildup

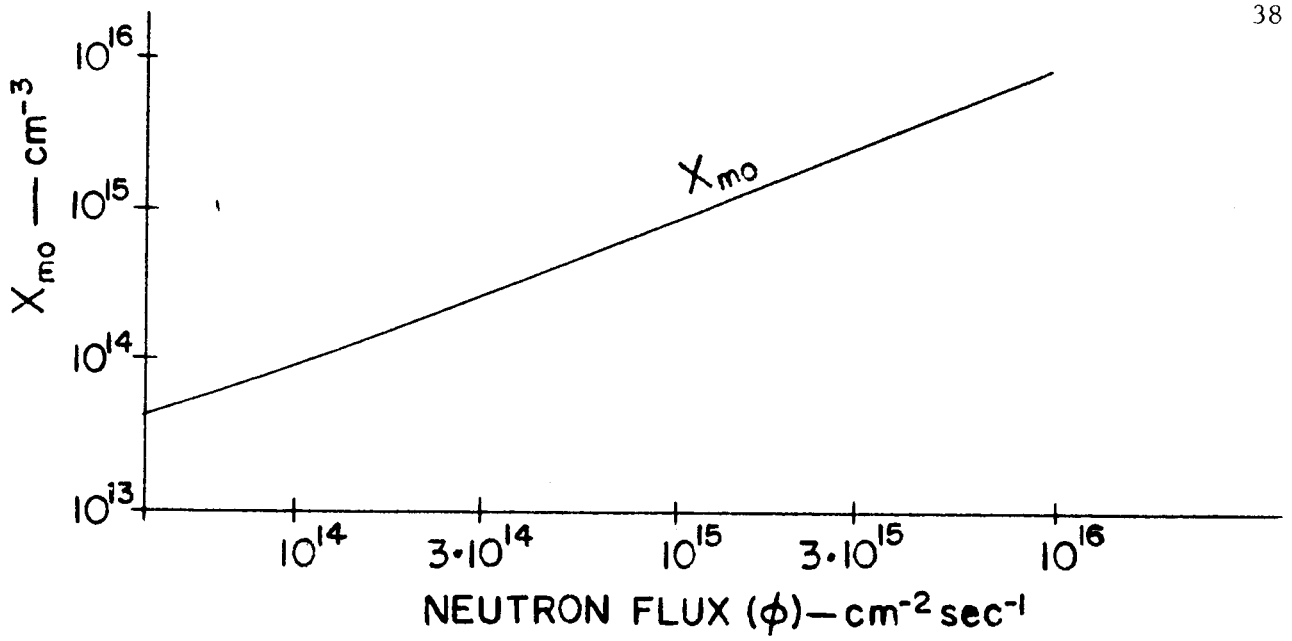


Fig. 2. Maximum Xenon Concentration after Shutdown

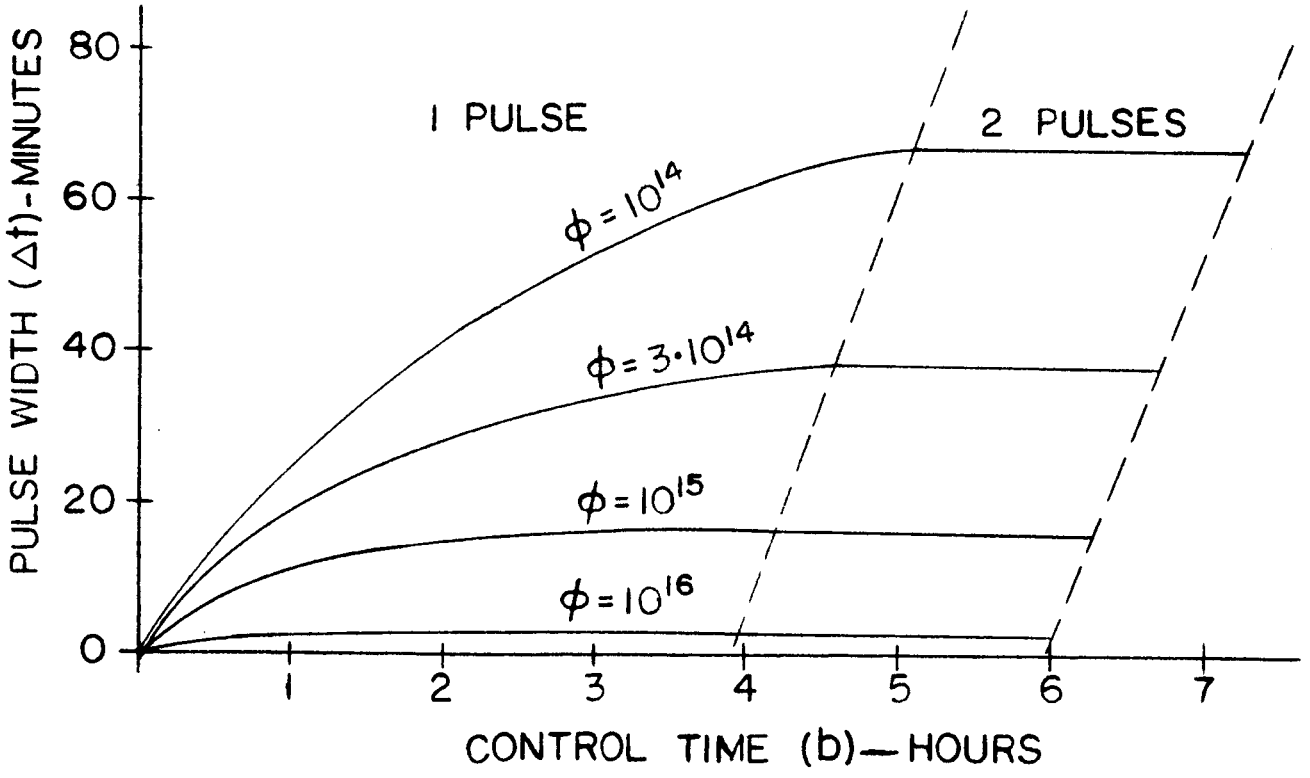
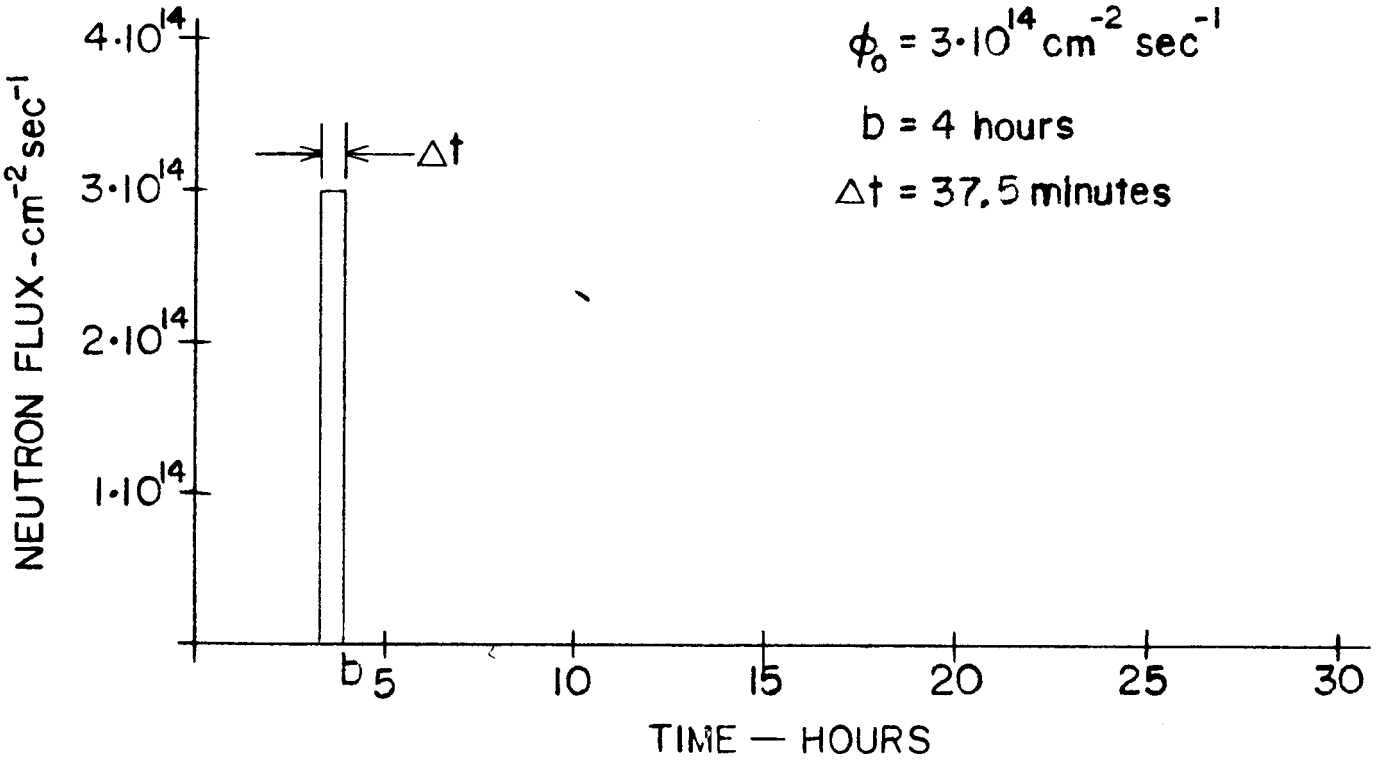


Fig. 4. Control Program for One Pulse Control



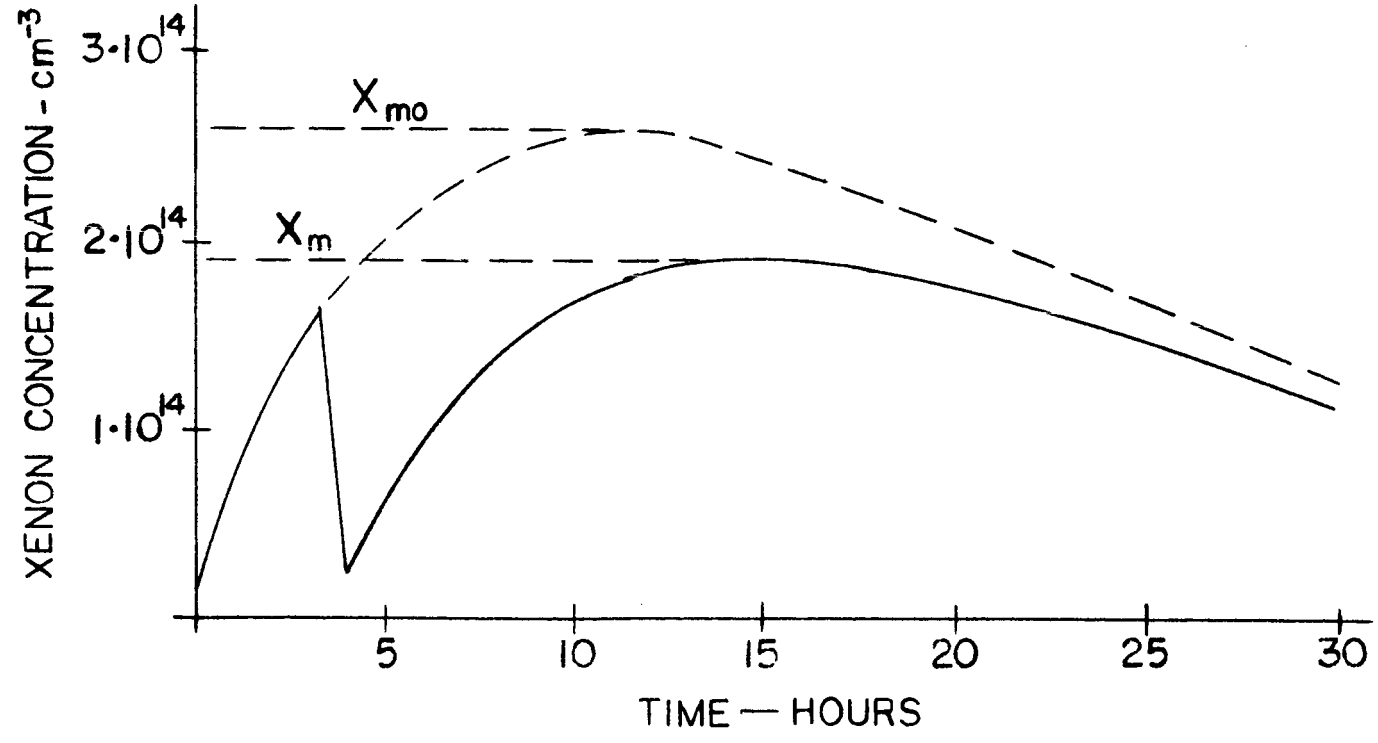
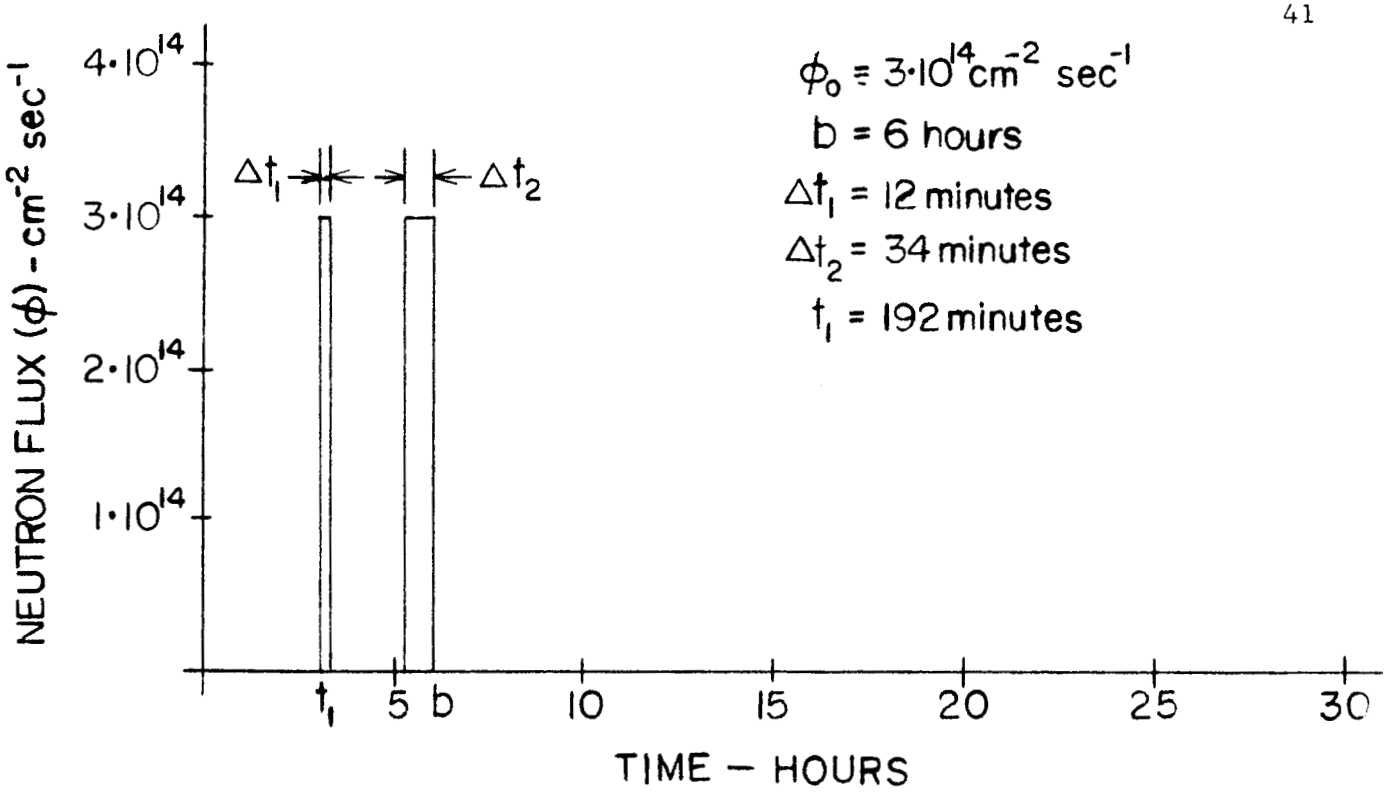


Fig. 3. One Pulse Control.

second pulse must appear before the terminal pulse in order to reduce the first peak.

When the control time extends into the two-pulse region, the pulse width for the pulse terminating at $t = b$ remains constant and a second pulse appears at $t = t_1$, as indicated in Fig. 5. As the control time interval b increases, the width of the pulse beginning at t_1, increases, t_1, remaining constant independent of the control time. It is noted that while the width of the pulse is well defined, the maximum value of the xenon concentration is relatively insensitive to the location of the first pulse. Variations in t_1, as large as $\frac{b}{10}$ produces very little change in the maximum xenon concentration. The optimal program in the two-pulse control region is described by three characteristic numbers: t_1, the time of occurrence of the first pulse; $\Delta 6t_1$, the width of the first pulse; and Δt_2, the width of the terminating pulse. A plot of these characteristic numbers are shown in Fig. 6. Knowing the value of the steady state flux at the time of shutdown and given a control time b, which falls into the two-pulse control region, the optimal shutdown program can be determined from Fig. 6.

Calculations based on the original problem state, even though they lose their physical meaning in this region can be carried out and serve as guidelines as to the best that may be accomplished in the region. It is clear from Fig. 1 that this is the case for the two-pulse control region. These guidelines are shown as dashed lines on Fig. 1. If the two-pulse control is extended beyond the two-pulse control region, the lines as shown in Fig. 1 curve away from those calculated on the premise of the original problem statement. It is reasonable to assume that beyond this point three-pulse control is needed. There is no proof as to how close it is possible



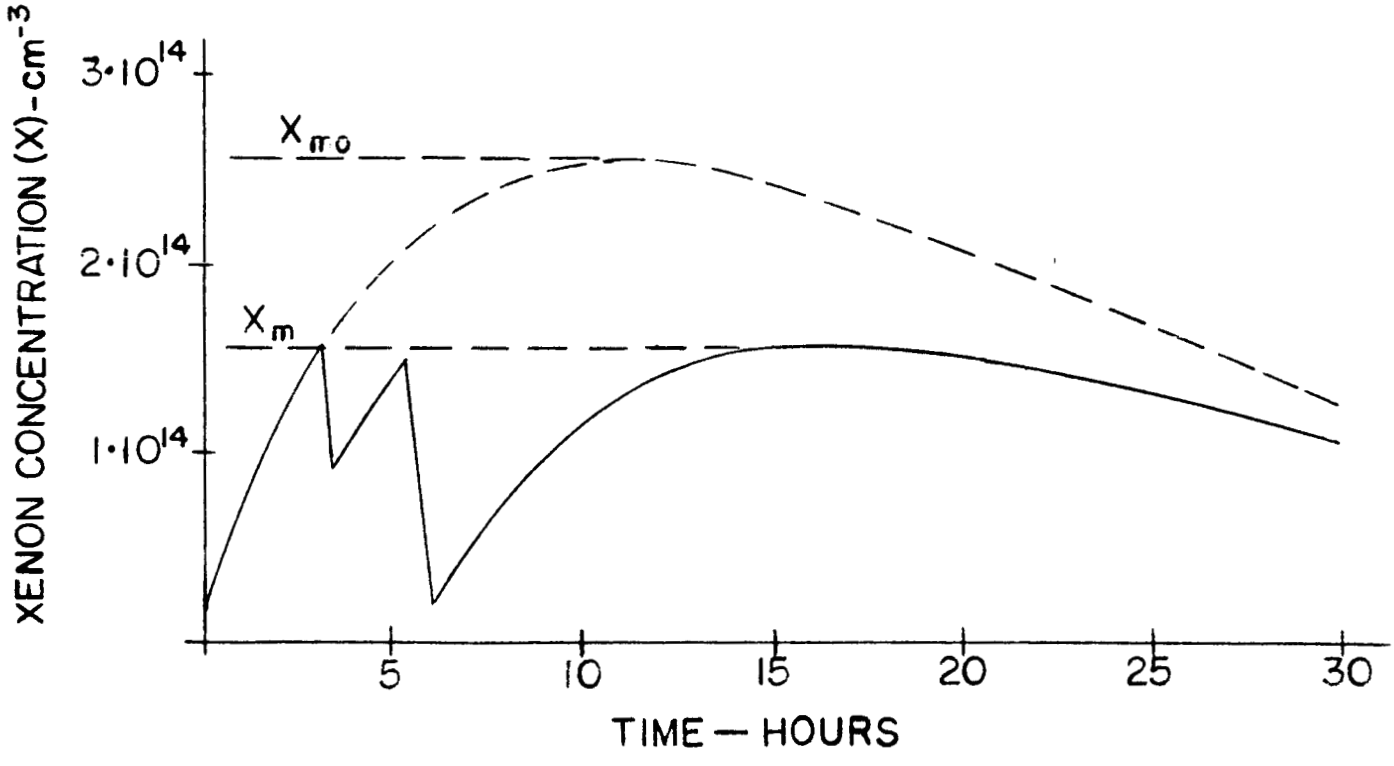
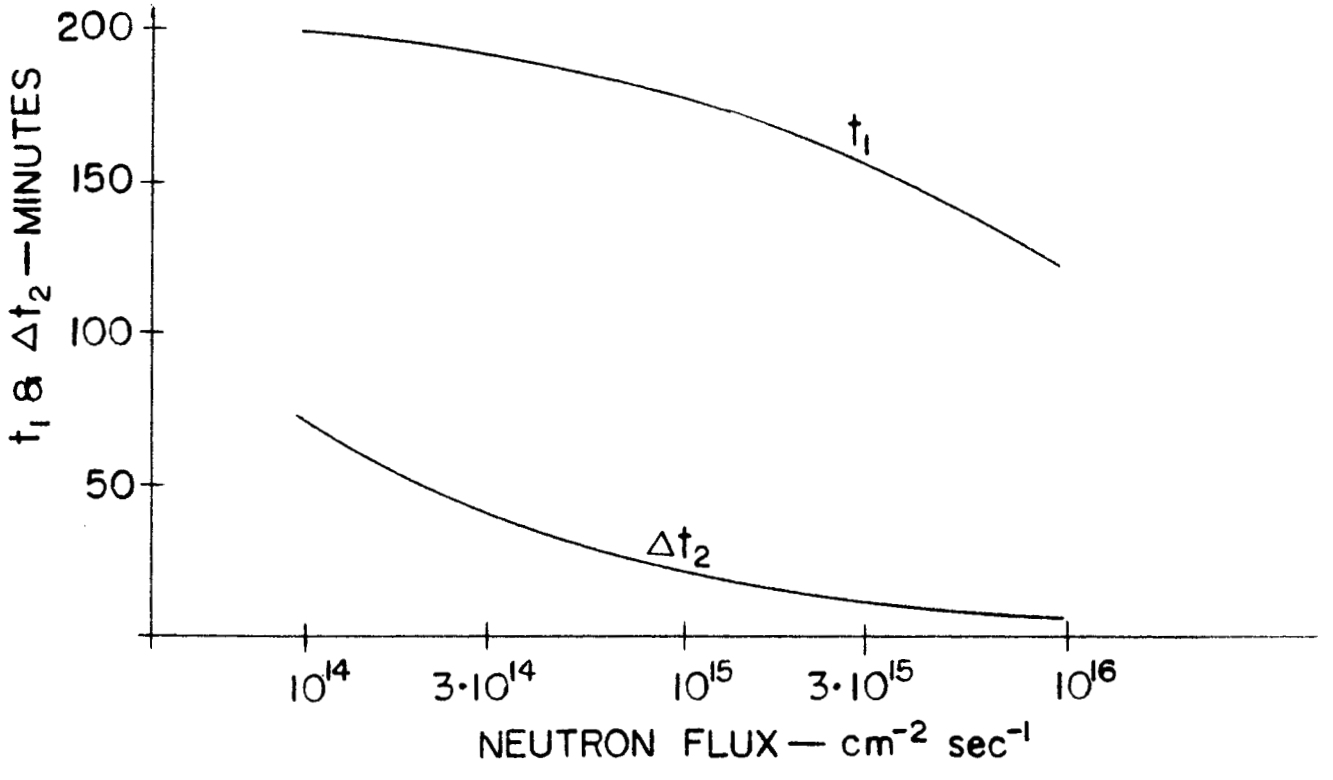


Fig. 5. Two Pulse Control.



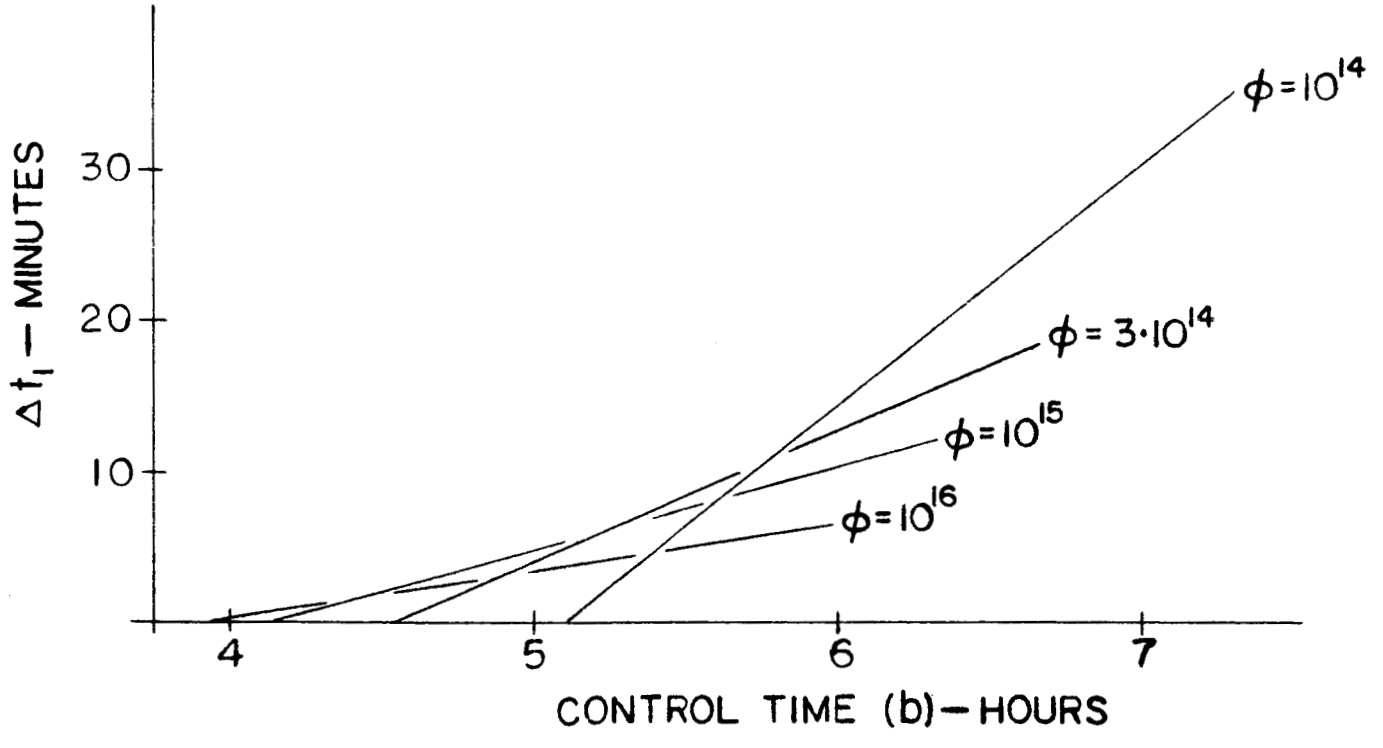


Fig. 6. Control Program for Two Pulse Control

to come to these optimum guidelines. It is conjectured that three, four and larger number of control pulses will result in further reduction in the maximum value of the xenon concentration for large values of control time.

V. Discussion

For the reactor model assumed it is clear that considerable reduction can be realized in the after shutdown buildup of xenon poisoning. The amount of reduction in xenon concentration depends on two parameters, the maximum flux level at the time of shutdown and the available control time. For high flux levels the fractional reduction in the xenon poisoning is greater for a given control time than that obtained at lower flux levels. Since the maximum value of the xenon concentration increases almost linearally with the flux level the reduction in xenon concentration is more dramatic for the high flux levels. For example, a control time of b = 6 hours, the fractional reduction is 27.7%, 36.6% and 44.5% corresponding to flux levels of 10^{14}, 3×10^{14} and 10^{16}, respectively. The values of the maximum xenon concentrations for these flux levels are 9.1×10^{13}, 2.6×10^{14} and 8.4×10^{15}, respectively. For U_{235} fuel, the relationship between poisoning and xenon concentration is given by

$$P = \frac{X\sigma_2}{\Sigma_t} = \frac{0.83\sigma_2}{\Sigma_f} X = 2.49 \times 10^{-15} X \qquad (22)$$

The equivalent reactivity is

$$\rho = fP \qquad (23)$$

where f is the thermal utilization factor for the reactor without poisoning. Using a value of f = 0.8, the reduction in terms of reactivity is

$$\Delta\rho = 2 \times 10^{-15} X_{mo} \left(1 - \frac{X_m}{X_{mo}}\right) \qquad (24)$$

For flux levels of 10^{14}, 3×10^{14}, and 10^{16} the percent reduction in reactivity that can be accomplished with optimal two-pulse control is 18, 52 and 1,680.

In all the cases presented above, it was assumed that the maximum control pulse level is equal to the equilibrium flux level at the time of shutdown. Since the pulse width is fairly small in many cases, it would seem reasonable to permit overloads for these short periods of time. This was done at the steady state flux level of 3×10^{14} by permitting the maximum value of the flux to be 4, 5, and 6×10^{14}. While the pulse width decreased appreciably, to about one-half at $\phi_{max} = 2\phi_o$, the reduction in xenon poisoning improved only slightly, changing from 25.5% to 26.1% for $\phi_{max} = 2\phi_o$ at $b = 4$ hours.

In deriving optimum shutdown programs, it was assumed that the flux could undergo step changes. In reality, reactors have period limitations and fast startups and shutdowns have to be avoided. To estimate the effect of period limitations, the example presented in Fig. 5 was worked out with a constraint on the maximum rate of change of flux, requiring 10 minutes to reach its peak value from startup and 10 minutes to return to zero. During this time the flux was assumed to change linearally for simplicity. This last assumption is more restrictive than the real case where the flux rises in an almost exponential manner. The effect of this distorted control pulse was essentially negligible and was less than one percent of that obtained using rectangular pulses.

Optimal After-Heat Removal

I. Introduction

When a nuclear reactor is shutdown, appreciable energy continues to be released due to the emission of beta and gamma rays from radioactive fission products. The rate at which energy or heat is released depends upon the reactor power level and period of operation prior to shutdown. For reactors operating at high-power levels, the after-shutdown heat generation can reach a level such that if not removed, will cause damage to the reactor. The after-shutdown heat removal is a problem in the operation of nuclear rocket engines. In the nuclear rocket engine the propellant serves as the coolant, and therefore, a certain amount of propellant must be reserved for the removal of heat generated after shutdown. The purpose of the investigation currently in progress is to determine optimum after-heat removal programs which will prevent overheating of the nuclear rocket engine, while using the smallest quantity of propellant.

II. Heat Transfer Model

An exact description of the temperature distribution and heat transfer within a nuclear rocket engine is quite involved. In the initial phase of this investigation a simplified heat transfer model which is believed to give a fairly satisfactory description of the system, is being used. More realistic models will be used after a greater insight into the problem has been obtained from the simplified model.

It is assumed that the heat transfer characteristics of the system can be described by a simple heat exchanger, based on the following assumptions:

a. There is no axial heat conduction.

b. The temperature of the fuel is uniform in the radial direction.

c. Heat is generated uniformly in a fuel element independent of position.

d. The temperature difference between the fuel and coolant $(T - T_c)$ is space independent and a function of time only.

Under these assumptions, the heat balance equation for a section dx of the reactor is:

$$QAdx = Mc_R \frac{dT}{dt} Adx + hpdx\ (T - T_c) \qquad (1)$$

where: Q = heat generation per unit volume per second

A = cross sectional area of the fuel

M = mass of fuel per unit volume

c_R = heat capacity of fuel material

T = temperature of fuel, a function of x only

h = heat transfer coefficient from fuel to coolant

p = wetted perimeter of coolant channel

T_c = coolant temperature, function of x only

If it is assumed that the coolant does not change phase in the reactor core, then all the heat transferred to the coolant goes in to increasing its entalpy

$$wc_p(T_c - T_i) = \int_o^x h_p\ (T - T_c)\ dx \qquad (2)$$

where: w = mass flow rate of coolant

c_p = heat capacity of coolant

T_i = coolant inlet temperature

Since $(T - T_c)$ was assumed independent of x it follows that

$$wc_p(T_c - T_i) = h_p(T - T_c)x \tag{3}$$

or

$$T_c = \frac{wc_pT_i}{wc_p + hpx} + \frac{hpxT}{wc_p + hpx} \tag{4}$$

Substituting equation (4) into equation (1), and rearranging terms, results in:

$$\frac{dT}{dt} = \frac{Q}{Mc_R} - \frac{1}{AMc_R} \frac{T - T_i}{\frac{1}{ph} + \frac{x}{wc_p}} \tag{5}$$

For the model given, the temperature is always the highest at $x = \ell$, the reactor outlet. Therefore, $T(x,t)$ can be replaced by $T(\ell,t) = T(t)$. If the coolant is a gas, the heat transfer coefficient is given by the familiar relationship:

$$\frac{hD}{k} = 0.027\ Re^{0.8}\ Pr^{0.33} \left(\frac{\mu}{\mu_w}\right)^{0.14} \tag{6}$$

where: D = hydraulic diameter of coolant channel

K = thermal conductivity of fuel

$Re = \frac{Dw}{\mu A_c}$ = Reynolds number

A_c = Cross-section of coolant channel

Pr = Prandtl number $= \frac{c_p\,\mu}{k}$

μ = viscosity at bulk fluid temperature

μ_w = viscosity at wall temperature

Therefore:

$$hp = 0.027\, k\pi \left(\frac{D}{\mu A_c}\right)^{0.8} w^{0.8}\, Pr^{0.33} \left(\frac{\mu}{\mu_w}\right)^{0.14} \qquad (7)$$

letting $k_1 = 0.027\, k\pi \left(\frac{D}{\mu A_c}\right)^{0.8} Pr^{0.33} \left(\frac{\mu}{\mu_w}\right)^{0.14}$

Equation (7) becomes:

$$hp = k_1 w^{0.8} \qquad (8)$$

If the reactor had been operating at a constant power P_o for a period of t_o hours, then the decay heat power t hours after shutdown is given by

$$P = 3.23 \cdot 10^{-3} P_o \left[t^{-0.2} - (t + t_o)^{-0.2} \right] \qquad (9)$$

The after-heat generation per unit volume per second is given by equation (10)

$$Q = 3.23 \cdot 10^{-3} \frac{P_o}{A\ell} \left[t^{-0.2} - (t + t_o)^{-0.2} \right] \qquad (10)$$

where $A\ell Q$ and P_o have the same dimensions.

Substituting equation (8) and (10) into equation (5) gives:

$$\frac{dT}{dt} = \frac{3.23 \cdot 10^{-3} P_o}{A\ell M c_R} \left[t^{-0.2} - (t + t_o)^{-0.2} \right] - \frac{c_p}{A\ell M c_R} \frac{T - T_i}{\frac{c_p}{\ell k_1 w^{0.8}} + \frac{1}{w}} \qquad (11)$$

letting

$$B = \frac{3.23 \cdot 10^{-3} P_o}{A \ell M c_R}, \quad C = \frac{c_p}{A \ell M c_R} \quad \text{and} \quad D = \frac{c_p}{\ell k_1}$$

then equation (11) becomes

$$\frac{dT}{dt} = B \left[t^{-0.2} - (t + t_o)^{-0.2} \right] - \frac{Cw(T - T_i)}{1 + Dw^{0.2}} \tag{12}$$

Equation (12) describes the system under consideration.

III. Solution of the Problem

The problem is to determine a program for the coolant flow rate, in the time interval $0 < t < t_1$, such that:

a. $T \leq T_{max}$ for all $0 < t < t_1$

b. The integral $\int_o^{t_1} w dt$ be as small as possible.

The problem can be divided into two separtate regions. When the system operates in the region $T < T_{max}$ the performance index is given by

$$S = \int_o^{t_1} w dt \tag{13}$$

With no additional restrictions the obvious result is $w = 0$ throughout the region. The temperature change in this region is given by:

$$\frac{dT}{dt} = B \left[t^{-0.2} - (t + t_o)^{-0.2} \right] \tag{14}$$

or

$$T = B \int_o^t \left[t^{-0.2} - (t + t_o)^{-0.2} \right] dt + T_o \tag{15}$$

where $T_o = T(o)$ in the given initial condition. The other possibility is for the system to operate on its boundary $T = T_{max}$. To apply the Maximum Principle to this region of operation, it must be stated in its modified form as given by Theorem 22 of Reference 3. For the simplified model given by equation (12), and with the restriction $T = T_{max}$, the problem can be solved without the use of the Maximum Principle. The solution is:

$$\frac{dT}{dt} = 0 = B\left[t^{-0.2} - (t + t_o)^{-0.2}\right] - \frac{Cw(T_{max} - T_i)}{1 + Dw^{0.2}} \qquad (16)$$

The optimal solution for the time interval $0 < t < t_1$ can be any combination of the above solutions. From physical considerations it is obvious that the most efficient heat removal is obtained when the outlet temperature of the coolant is at its maximum. For the model given by equation (12), it is equivalent to $T = T_{max}$. Therefore, the optimal coolant flow rate in the sub-interval $0 < t < t'$ is zero. The coolant flow rate in the sub-interval $t' < t < t_1$ is given by equation (16). The value of t' can be calculated from equation (15) by letting $T = T_{max}$. The corresponding history of the temperature is given by equation (15) for the interval $0 < t < t'$ and is equal to $T = T_{max}$ for the interval $t' < t < t_1$.

For the simplified model assumed here, the solution is fairly obvious and the Maximum Principle is not needed. During the next report period more realistic models for the system will be used. It is doubtful that the solutions using more complex models will be quite as obvious.

REFERENCES

1. Ash, M., et al, "On Control of Reactor Shutdown Involving Minimal Xenon Poisoning," Nuclear Science and Engineering 6, 152-156 (1959).

2. Fresdall, J. R. and Babb, A. L., "Xenon-135 Transients Resulting from Time-Varying Shutdowns of Thermal Reactors," Trans. Am. Nuclear Soc., 4, 2 (1961).

3. Pontryagin, L. S., et al, "The Mathematical Theory of Optimal Processer, Interscience Publishers, New York (1962).